

Exam review problems

1. $x' - x = e^t$

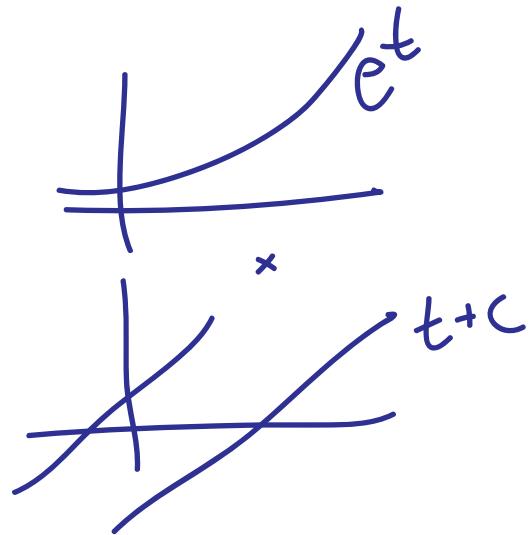
$$e^{-t} x' - e^{-t} x = 1$$

$$e^{-t} x = t + c$$

$$x = (t + c)e^t$$

$$x(t) \rightarrow +\infty \quad (\text{a})$$

for any c .



2. $y' = -\frac{x}{y}$, $y(0) = -2$

$$yy' = -x$$

$$\left(\frac{1}{2}y^2\right)' = -x$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$y = \pm \sqrt{D - x^2}$$

$$y(0) = -2 \text{ so choose } - \text{ and } D = 4.$$

(c)

$$3. \quad y'' + 16\pi^2 y = f(t)$$

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} B_n \sin \frac{2\pi n t}{T}$$

$$y_h(t) = C_1 \sin 4\pi t + C_2 \cos 4\pi t$$

We only have a conflict between

$y_h(t)$ and $f(t)$ if $\frac{2\pi n}{T} = 4\pi$ for

some n . i.e. $n = 2T$. No problem.

$$y_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T} \quad (c)$$

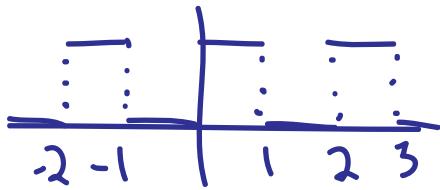
Note A_0, A_n, B_n are the Fourier coefficients of $f(t)$ and a_0, a_n and b_n are the (unknown) Fourier coefficients of y_p .

$$a_0 = \frac{A_0}{16\pi^2}, \quad a_n = \frac{A_n}{\frac{16\pi^2 - 4\pi^2 n^2}{T^2}}$$

$$b_n = \frac{B_n}{\frac{16\pi^2 - 4\pi^2 n^2}{T^2}}.$$

$$4. y'' + 81y = f(t) = \begin{cases} 1 & n < t \leq n+1 \\ 0 & n+1 < t \leq n+2 \end{cases}$$

for $n = \dots -4, -2, 0, 2, 4, \dots$



Note that

$$f(t) = \frac{1}{2} + \text{odd function}$$

$$\text{so } f(t) = \frac{a_0}{2} + \sum b_n \sin n\pi t \quad \text{and } a_0 = 1.$$

$$\begin{aligned} b_n &= \int_0^1 \sin n\pi t dt = -\frac{1}{n\pi} \cos n\pi t \Big|_0^1 \\ &= \frac{1}{n\pi} (1 - (-1)^n) \\ &= \frac{2}{\pi}, 0, \frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0 \dots \end{aligned}$$

$$y_p = B_n \sin n\pi t$$

$$y_p'' = -n^2 \pi^2 B_n \sin n\pi t$$

$$y_p'' + 81y_p = (81 - n^2 \pi^2) B_n \sin n\pi t = b_n \sin n\pi t$$

$$B_n = \frac{b_n}{81 - n^2 \pi^2} = \frac{\frac{2}{\pi}}{\frac{81 - \pi^2}{(\approx 72)}}, 0,$$

$n=3$

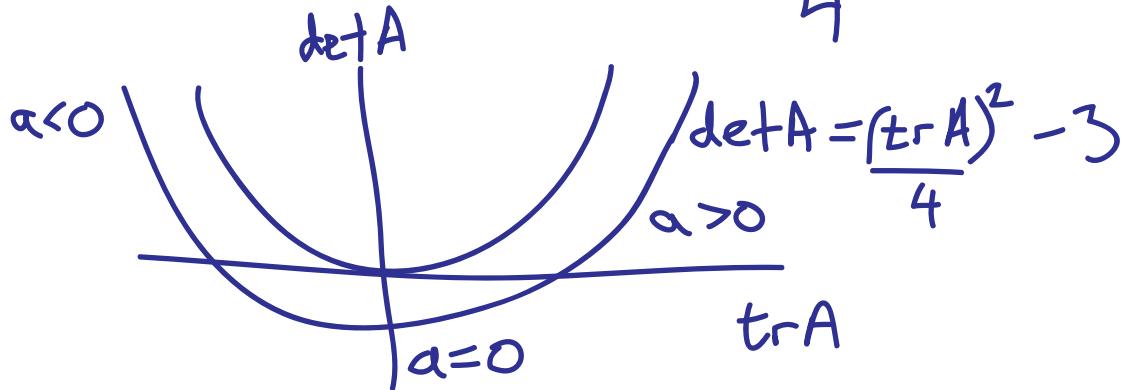
$\frac{\frac{2}{3\pi}}{\frac{81 - 9\pi^2}{(\approx \text{small})}}$

$\frac{\frac{2}{5\pi}}{\frac{81 - 25\pi^2}{(\approx -94)}}$

$$5. A = \begin{pmatrix} a & 3 \\ 1 & a \end{pmatrix}$$

$$\operatorname{tr} A = 2a$$

$$\det A = a^2 - 3 = \frac{(\operatorname{tr} A)^2 - 3}{4}$$



stable node — saddle — unstable node (c)

$$6. Y(s) = \frac{s+3}{(s+1)^2 + 4} = \frac{s+1}{(s+1)^2 + 4} + \frac{2}{(s+1)^2 + 4}$$

$$y(t) = e^{-t} (\cos 2t + \sin 2t)$$

$$= \sqrt{2} e^{-t} \left(\frac{1}{\sqrt{2}} \cos 2t - \left(-\frac{1}{\sqrt{2}} \right) \sin 2t \right)$$

$$= \sqrt{2} e^{-t} \cos \left(2t - \frac{\pi}{4} \right) \quad \left[\begin{array}{l} \cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \\ \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} \end{array} \right]$$

$$= \sqrt{2} e^{-t} \cos \left(2(t - \frac{\pi}{8}) \right) \quad (a)$$

$$7. \quad u_t = 16u_{xx}$$

$$u(x,t) = e^{\lambda_n t} \sin \sqrt{\frac{-\lambda_n}{16}} x$$

$$u(0,t) = 0 \quad \checkmark$$

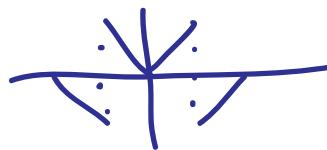
$$u_x(4,t) = \sqrt{\frac{-\lambda_n}{16}} e^{\lambda_n t} \cos \sqrt{\frac{-\lambda_n}{16}} \cdot 4 = 0$$

$$\cos \sqrt{\frac{-\lambda_n}{16}} \cdot 4 = 0 \quad \text{when } \sqrt{-\lambda_n} = \frac{n\pi}{2}, n \text{ odd}$$

$$\lambda_n = -\frac{n^2\pi^2}{4}$$

$$u_n(x,t) = e^{-\frac{n^2\pi^2}{4}t} \sin \frac{n\pi x}{8} \quad (c)$$

8. Extend $f(x)$ as even about $x=0$ and

as odd about $x=1$.  (c)

$$9. \quad -D u_x(0,t) = 5 \text{ g/min} \quad D = 2 \text{ m}^2/\text{min}$$

$$u(3,t) = 0$$

$$u_{ss}(x) = ax + b = -\frac{5}{2}x + \frac{15}{2} \quad (d)$$

$$\frac{du_{ss}}{dx} = a = -\frac{5}{2}$$

$$u_{ss}(3) = -\frac{15}{2} + b = 0 \Rightarrow b = \frac{15}{2}$$

$$10. \quad y''' + 2y'' = t + e^{-2t}$$

$$y = e^{rt} \Rightarrow r^3 + 2r^2 = 0$$

$$r^2(r+2) = 0$$

$$r=0 \quad (\text{repeated root}) \quad y = 1, t$$

$$r=-2 \quad y = e^{-2t}$$

$$y_h(t) = C_1 + C_2 t + C_3 e^{-2t}$$

$$y_p(t) = (At + B)t^2 + Cte^{-2t}$$

(d)

11. (a)

12. (a)