Today

- Fourier Series examples
- The Diffusion (Heat) Equation derivation
- Solving the Diffusion equation subject to BCs and IC using FS

This pdf is also posted on the lecture slides page.

Deriving the FS coefficient formulae

Define the dot product for periodic functions (with period P)

$$f(x)\circ g(x) = \int_{\substack{\text{one}\\\text{period}}} f(x)\cdot g(x)\,dx = \int_{-P/2}^{P/2} f(x)g(x)\,dx$$

Let $v_n(x) = \cos\left(\frac{2\pi n x}{P}\right)$, $w_n(x) = \sin\left(\frac{2\pi n x}{P}\right)$, $v_0(x) = 1$. $(n = 1, 2, \cdots)$

Recall (or calculate for yourself) that

$v_0(x)\circ v_0(x) = P$, $v_m(x)\circ v_n(x) = 0$ for $m \neq n$, $v_n(x)\circ v_n(x) = P/2$

$w_m(x)\circ v_n(x) = 0$, $w_m(x)\circ w_n(x) = 0$ for $m \neq n$, $w_n(x)\circ w_n(x) = P/2$

Suppose $f(x)$ can be represented exactly as a FS. Thus

$$f(x) = A_0 v_0(x) + \sum_{m=1}^{\infty} a_m v_m(x) + \sum_{m=1}^{\infty} b_m v_m(x).$$

Find its FS coefficients. As with vectors, use "$\circ$" to find A_0, a_n, b_n.

To find A_0,

$$f(x)\circ v_0(x) = A_0 v_0(x)\circ v_0(x) + \sum_{m=1}^{\infty} a_m v_m(x)\circ v_0(x) + \sum_{m=1}^{\infty} b_m w_m(x)\circ v_0(x) = A_0\cdot P$$

Thus, $A_0 = \frac{1}{P} f(x)\circ v_0(x) = \frac{1}{P}\int_{-P/2}^{P/2} f(x)\,dx.$

To find a_n,

$$f(x)\circ v_n(x) = A_0 v_0(x)\circ v_n(x) + \sum_{m=1}^{\infty} a_m v_m(x)\circ v_n(x) + \sum_{m=1}^{\infty} b_m w_m(x)\circ v_n(x) = a_n \underbrace{v_n(x)\circ v_n(x)}_{P/2}$$

Thus, $a_n = \frac{2}{P} f(x)\circ v_n(x) = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\frac{2n\pi x}{P}\,dx.$

Similarly, $b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\frac{2n\pi x}{P}\,dx$

In many cases, we will have $P = 2L$ (but not always!) so

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier series

- Calculate the coefficients of the Fourier series of a function:

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

$$v_0(x) = 1$$
$$v_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$
$$w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$f_{FS}(x) = \frac{a_0}{2} v_0(x) + a_1 v_1(x) + a_2 v_2(x) + \cdots$$
$$+ b_1 w_1(x) + b_2 w_2(x) + \cdots$$

$$f_{FS}(x) \circ v_n(x) = \frac{a_0}{2} \cancelto{0}{v_0(x) \circ v_n(x)} + a_1 \cancelto{0}{v_1(x) \circ v_n(x)} + a_2 \cancelto{0}{v_2(x) \circ v_n(x)} + \cdots$$
$$+ b_1 \cancelto{0}{w_1(x) \circ v_n(x)} + b_2 \cancelto{0}{w_2(x) \circ v_n(x)} + \cdots$$

$$= a_n v_n(x) \circ v_n(x) = a_n L$$

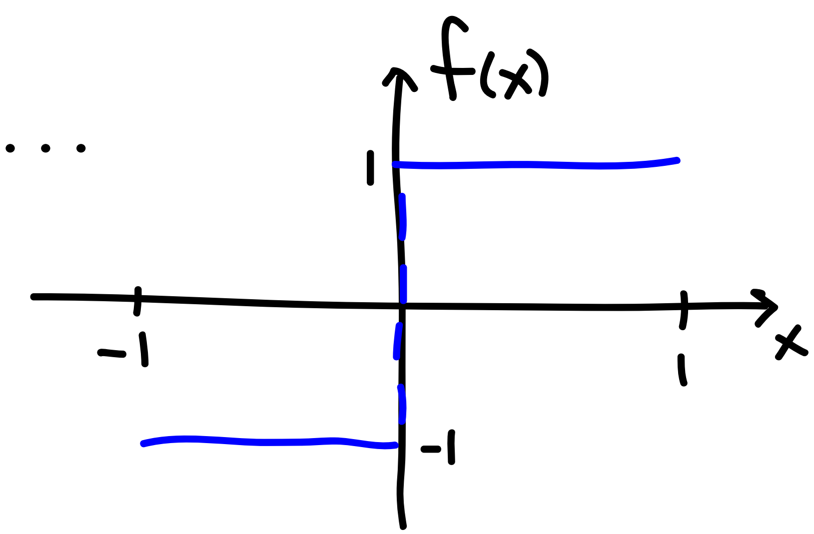
$$a_n = \frac{1}{L} \int_{-L}^{L} f_{FS}(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

Fourier series

- Find the Fourier series for $f(x) = 2u_0(x)-1$ on the interval [-1,1].

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$



- Our hope is that $f(x) = f_{FS}(x)$ so we calculate coefficients as if they were equal:

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

A_0 is the average value of f(x)!

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

- To simplify formulas, usually define

$$a_0 = 2A_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

Fourier series

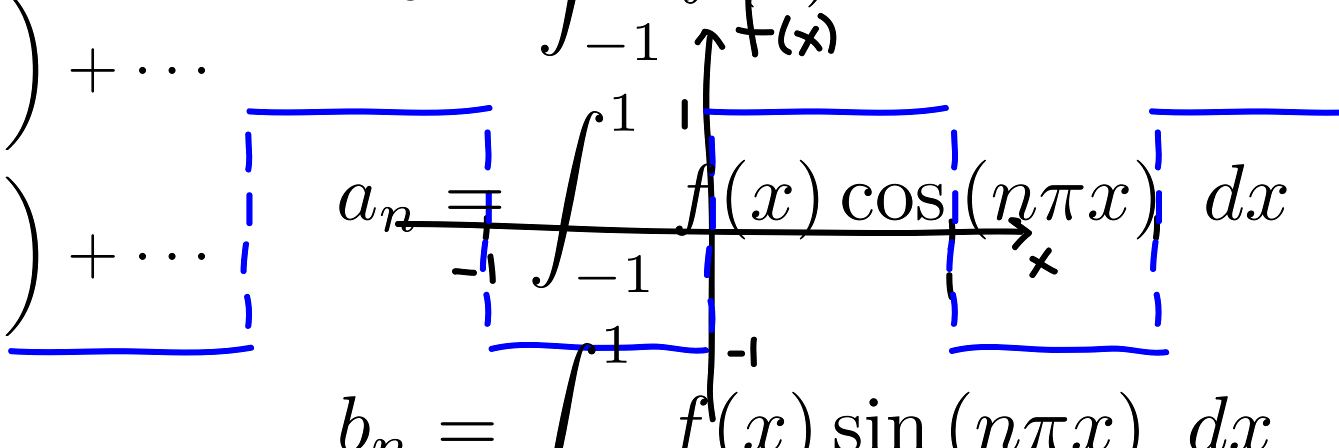
- Calculate the coefficients.

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

$$a_0 = \int_{-1}^{1} f(x)\, dx$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx$$



$a_0 =$ / $b_n =$

(A) 0

(B) $\frac{1}{\pi}$ / (B) $\frac{2}{n\pi}$

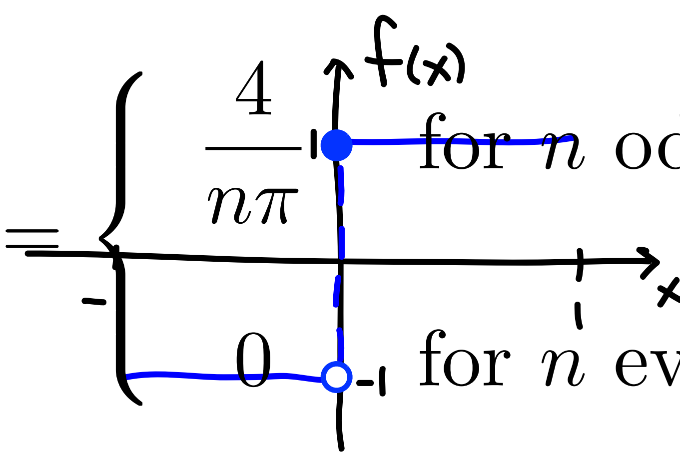
(C) undefined

(C) $\frac{4(-1)^n}{n\pi}$

(D) $\frac{1}{n\pi}$

(D) $\frac{2(1-(-1)^n)}{n\pi}$

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$



$$f_{FS}(x) = \frac{4}{\pi} \sin\left(\frac{\pi x}{L}\right) + \frac{4}{3\pi} \sin\left(\frac{3\pi x}{L}\right) + \frac{4}{5\pi} \sin\left(\frac{5\pi x}{L}\right) + \cdots$$

https://www.desmos.com/calculator/tlvtikmi0y

Does f(x) = f_{FS}(x) for all x?

Problems at jumps! x=-1, 0, 1

Fourier series

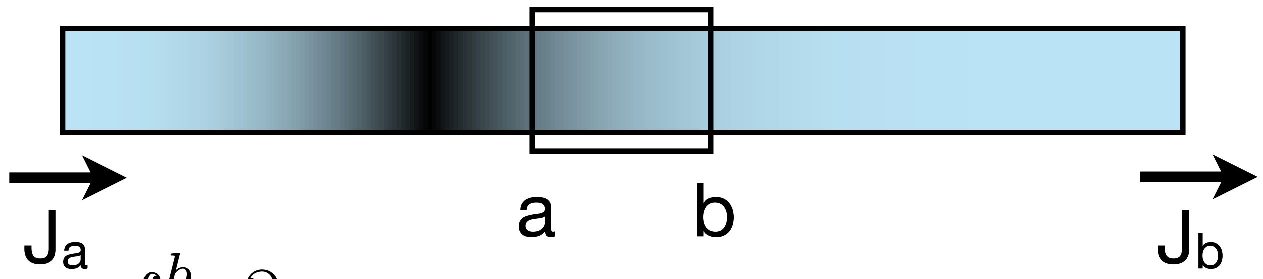
- **Theorem** Suppose f and f' are piecewise continuous on [-L,L] and periodic beyond that interval. Then f(x) = $f_{FS}(x)$ at all points at which f is continuous. Furthermore, at points of discontinuity, $f_{FS}(x)$ takes the value of the midpoint of the jump. That is,

$$f_{FS}(x) = \frac{f(x^+) + f(x^-)}{2}$$

The Diffusion Equation

c(x,t) is linear mass density of ink in a long narrow tube.

$$Q_{ab}(t) = \int_a^b c(x,t)\ dx$$



$$\frac{dQ_{ab}}{dt}(t) = \frac{d}{dt}\int_a^b c(x,t)\ dx\ = \int_a^b \frac{\partial}{\partial t} c(x,t)\ dx$$

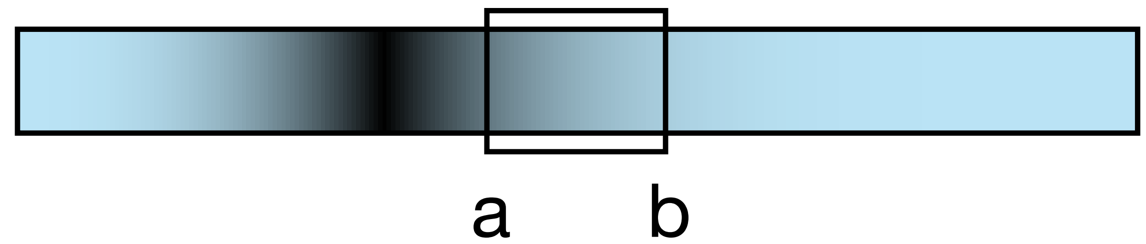
Define the flux J_a to be the amount of mass crossing the line x=a per unit of time (particles moving right count as positive flux) .

In that case, the change of Q inside the a-b box can also be counted watching flux, that is, flux at a - flux at b:

$$\frac{dQ_{ab}}{dt}(t) = \ -J_b + J_a$$

The Diffusion Equation

$$Q_{ab}(t) = \int_a^b c(x,t)\ dx$$



$$\frac{dQ_{ab}}{dt}(t) = \frac{d}{dt}\int_a^b c(x,t)\ dx \ = \int_a^b \frac{\partial}{\partial t} c(x,t)\ dx$$

$$\frac{dQ_{ab}}{dt}(t) = \ -J_b + J_a$$

Need a model for flux. Let's consider simpler case first (not diffusion yet!)

If fluid in pipe is moving with velocity v, flux is vc: $J_a = vc(a,t)$

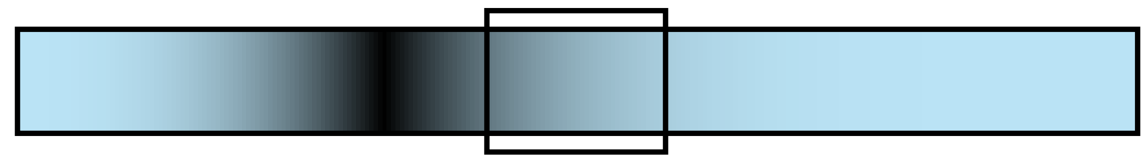
$$\frac{dQ_{ab}}{dt}(t) = -J_b + J_a \ = -vc(b,t) + vc(a,t) = - \ vc(x,t)\Big|_a^b \ = - \int_a^b v\frac{\partial c}{\partial x}\ dx$$

$$\int_a^b \frac{\partial}{\partial t} c(x,t)\ dx \ = - \int_a^b v\frac{\partial c}{\partial x}\ dx \ \Rightarrow \ \frac{\partial c}{\partial t} = -v\frac{\partial c}{\partial x}$$

Called Transport equation.

The Diffusion Equation

$$Q_{ab}(t) = \int_a^b c(x,t)\ dx$$



$$\frac{dQ_{ab}}{dt}(t) = \frac{d}{dt}\int_a^b c(x,t)\ dx\ = \int^b \frac{\partial}{\ } c(x,t)\ dx$$

$$\frac{dQ_{ab}}{dt}(t) = \ -J_b + J_a$$

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

Now lets consider diff ... from chemical potential but it also makes sense that for diffusion: $J_a = -D\ \left.\frac{\partial c}{\partial x}\right|_{x=a}$

$$\frac{dQ_{ab}}{dt}(t) = -J_b + J_a\ = D\ \left.\frac{\partial c}{\partial x}\right|_{x=b} - D\ \left.\frac{\partial c}{\partial x}\right|_{x=a} = D\ \left.\frac{\partial c}{\partial x}\right|_a^b$$

$$\int_a^b \frac{\partial}{\partial t}c(x,t)\ dx\ = \int_a^b D\frac{\partial^2 c}{\partial x^2}\ dx \quad \Rightarrow \quad \frac{\partial}{\partial t}c(x,t) = D\frac{\partial^2}{\partial x^2}c(x,t)$$

The Diffusion equation

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

- What does a steady state of the Diffusion equation look like?

$$0 = D\frac{d^2c}{dx^2}$$

$$c_{ss}(x) = Ax + B$$

The Diffusion Equation

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

Guess: $c(x,t) = ae^{bt}\sin(wx)$

$$\frac{\partial c}{\partial t} = abe^{bt}\sin(wx)$$

$$D\frac{\partial^2 c}{\partial x^2} = -Daw^2e^{bt}\sin(wx)$$

$$a = -Dw^2$$

$$c(x,t) = ae^{-w^2 Dt}\sin(wx)$$

Still need to determine a and w. Need to impose other conditions:

- A time derivative requires an initial condition c(x,0).
- Two space derivatives require two boundary conditions c(0,t) and c(L,t).