Today

- The geometry of homogeneous and nonhomogeneous matrix equations
- Solving nonhomogeneous equations
    - Method of undetermined coefficients

Tutorial poll

(A) Hand out worksheet on Friday, print and hand in during tutorial.

(B) Hand out worksheet during tutorial, hand in during Tuesday class.

Second order, linear, constant coeff, **non**homogeneous (3.5)

- Our next goal is to figure out how to find solutions to nonhomogeneous equations like this one:

$$y'' - 6y' + 8y = \sin(2t)$$

- But first, a bit more on the connections between matrix algebra and differential equations . . .

Some connections to linear (matrix) algebra

- An m×n matrix is a gizmo that takes an n-vector and returns an m-vector:

$$\overline{y} = A\overline{x}$$

- It is called a linear operator because it has the following properties:

$$A(c\overline{x}) = cA\overline{x}$$

$$A(\overline{x} + \overline{y}) = A\overline{x} + A\overline{y}$$

- Not all operators work on vectors. Derivative operators take a function and return a new function. For example,

$$z = L[y] = \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y$$

- This one is linear because

$$L[cy] = cL[y]$$

$$L[y + z] = L[y] + L[z]$$

Note: y, z are functions of t and c is a constant.

Some connections to linear (matrix) algebra

- A homogeneous matrix equation has the form

$$A\overline{x} = \overline{0}$$

- A non-homogeneous matrix equation has the form

$$A\overline{x} = \overline{b}$$

- A homogeneous differential equation has the form

$$L[y] = 0$$

- A non-homogeneous differential equation has the form

$$L[y] = g(t)$$

Solutions to homogeneous matrix equations

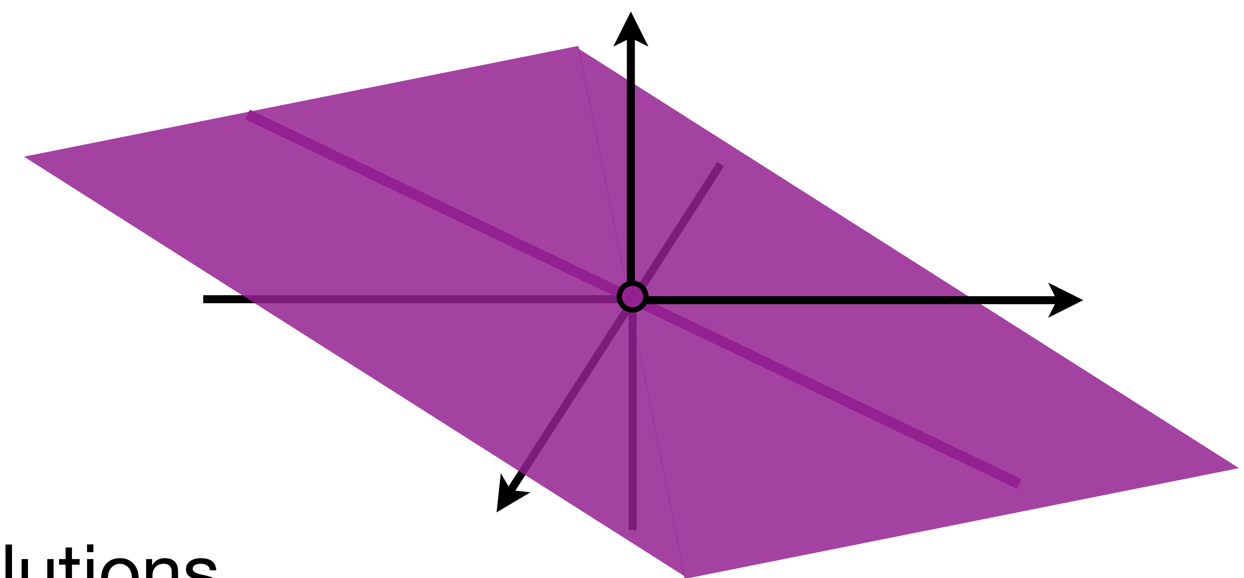
- The matrix equation $A\overline{x} = \overline{0}$ could have (depending on A)

(A) no solutions.

(B) exactly one solution.

(C) a one-parameter family of solutions.

(D) an n-parameter family of solutions.



Possibilities:

$$\overline{x} = \overline{0}$$

$$\overline{x} = C \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\overline{x} = C_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Choose the answer that is incorrect.

Solutions to homogeneous matrix equations

- Example 1. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

Each equation describes a plane.

- Row reduction gives

$$A \sim \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case, only two of them really matter.

- so $x_1 - \dfrac{1}{3}x_3 = 0$ and $x_2 + \dfrac{5}{3}x_3 = 0$ and x_3 can be whatever (because it doesn't have a leading one).

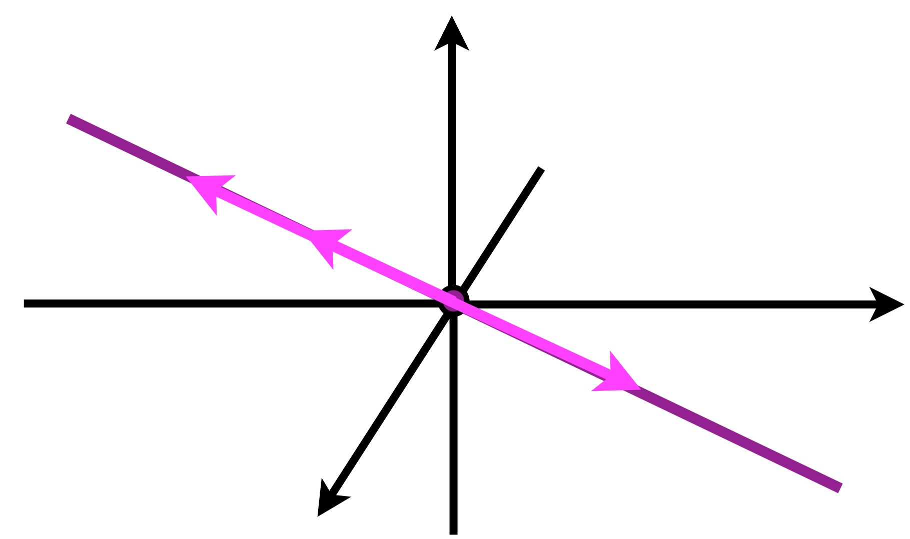
Solutions to homogeneous matrix equations

- Example 1. Solve the equation $A\overline{x} = \overline{0}$.
- so $x_1 - \frac{1}{3}x_3 = 0$ and $x_2 + \frac{5}{3}x_3 = 0$ and x_3 can be whatever.

$$x_1 = \frac{1}{3}x_3 \qquad x_1 = \frac{1}{3}C$$

$$x_2 = -\frac{5}{3}x_3 \qquad x_2 = -\frac{5}{3}C$$

$$x_3 = C$$



- Thus, the solution can be written as $\overline{x} = \frac{C}{3}\begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \\ \mathbf{3} \end{pmatrix}$.

Solutions to homogeneous matrix equations

- Example 2. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

- Row reduction gives

$$A \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- so $x_1 - 2x_2 + x_3 = 0$ and both x_2 and x_3 can be whatever.

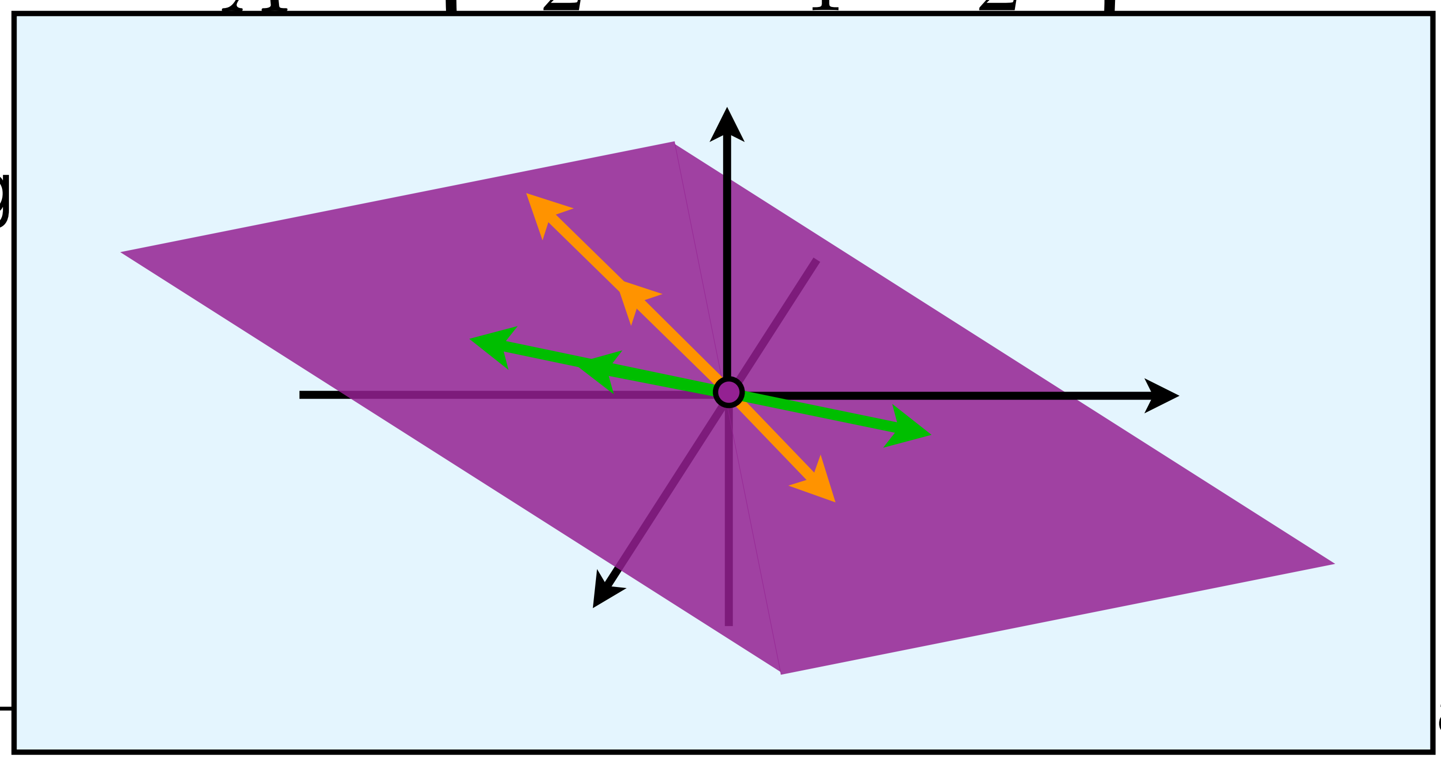
$$\overline{x} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solutions to homogeneous matrix equations

- Example 2. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{pmatrix}$$

- Row reduction g



- so $x_1 - 2x_2 +$ … atever.

$$\overline{x} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solutions to non-homogeneous matrix equations

- Example 3. Solve the equation $A\overline{x} = \overline{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

- Row reduction gives

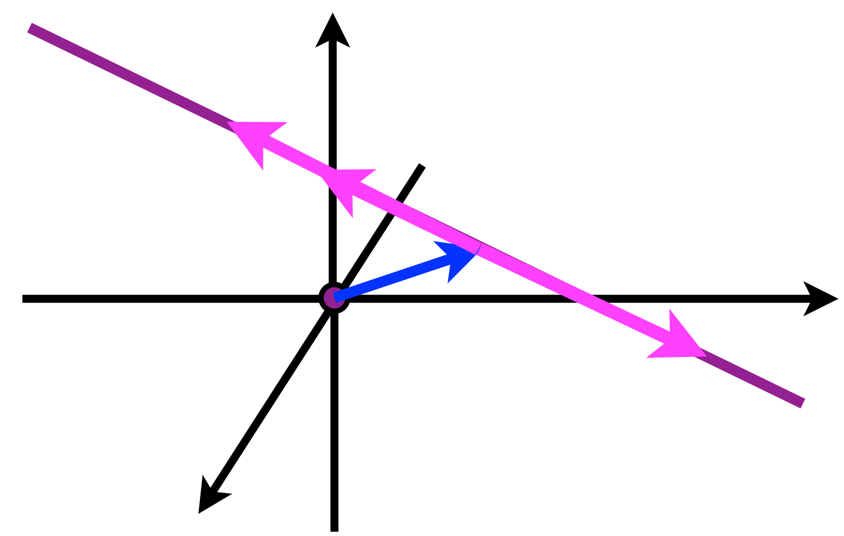
$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 5/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- so $x_1 - \frac{1}{3}x_3 = \frac{2}{3}$ and $x_2 + \frac{5}{3}x_3 = \frac{2}{3}$ and x_3 can be whatever.

Solutions to non-homogeneous matrix equations

- Example 3. Solve the equation $A\overline{x} = \overline{b}$.
- so $x_1 - \frac{1}{3}x_3 = \frac{2}{3}$ and $x_2 + \frac{5}{3}x_3 = \frac{2}{3}$ and x_3 can be whatever.

$$x_1 = \frac{1}{3}x_3 + \frac{2}{3} \qquad x_2 = -\frac{5}{3}x_3 + \frac{2}{3}$$



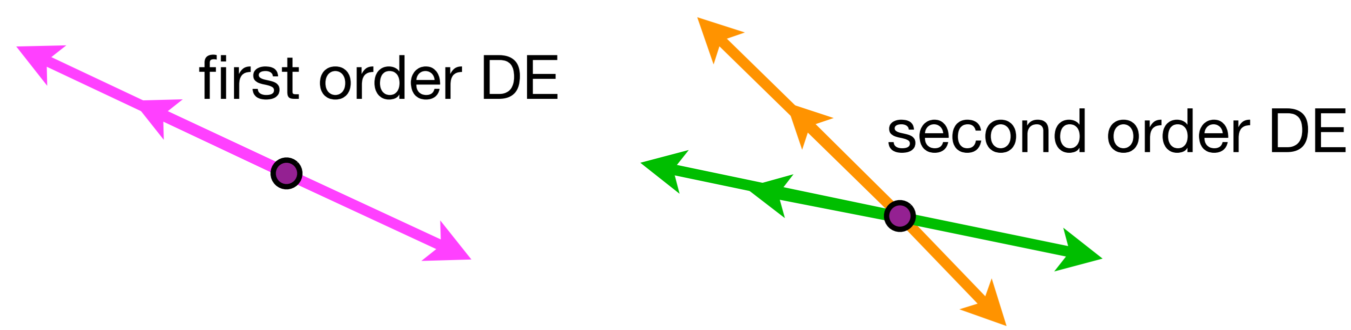
$$\overline{\boldsymbol{x}} = \frac{C}{3}\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \\ 0 \end{pmatrix}$$

the general solution to the homogeneous problem

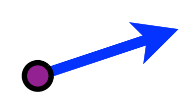
one particular solution to nonhomogeneous problem

Solutions to nonhomogeneous differential equations

- To solve a nonhomogeneous differential equation:
  1. Find the general solution to the associated homogeneous problem, $y_h(t)$.

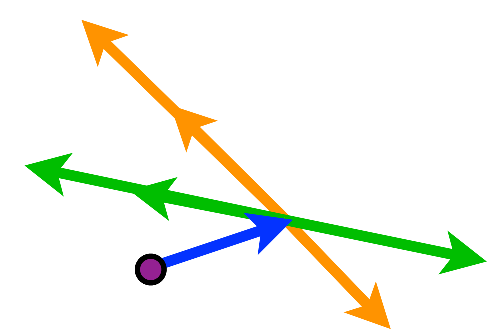


  2. Find a particular solution to the nonhomogeneous problem, $y_p(t)$.



  3. The general solution to the nonhomogeneous problem is their sum:

$$y = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$$



- For step 2, try “Method of undetermined coefficients”...

Method of undetermined coefficients

- Example 4. Define the operator $L[y] = y'' + 2y' - 3y$. Find the general solution to $L[y] = e^{2t}$. That is, $y'' + 2y' - 3y = e^{2t}$.
  - Step 1: Solve the associated homogeneous equation
$$y'' + 2y' - 3y = 0.$$
$$y_h(t) = C_1 e^{t} + C_2 e^{-3t}$$
  - Step 2: What do you have to plug in to $L[\ \cdot\ ]$ to get e^{2t} out?
    - Try $y_p(t) = Ae^{2t}$.

$$L[y_p(t)] = L[Ae^{2t}] =$$

(A) $5e^{2t}$ (C) $4e^{2t}$

★ (B) $5Ae^{2t}$ (D) $4Ae^{2t}$

    - A is an undetermined coefficient (until you determine it).

Method of undetermined coefficients

- Example 4. Define the operator $L[y] = y'' + 2y' - 3y$. Find the general solution to $L[y] = e^{2t}$. That is, $y'' + 2y' - 3y = e^{2t}$.
  - Summarizing:
    - We know that, for any C_1 and C_2,

$$L[C_1 e^t + C_2 e^{-3t}] = 0$$

    - We also know that

$$L[Ae^{2t}] = 5Ae^{2t}$$

    - Finally, by linearity, we know that

$$L[C_1 e^t + C_2 e^{-3t} + Ae^{2t}] = 0 + 5Ae^{2t}$$

    - So what's left to do to find our general solution? Pick A =?1/5.

Method of undetermined coefficients

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the solution to the associated homogeneous equation?

★ (A) $y_h(t) = C_1 e^{2t} + C_2 e^{-2t}$

(B) $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$

(C) $y_h(t) = C_1 e^{2t} + C_2 t e^{2t}$

(D) $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t) + e^t$

(E) Don't know.

Method of undetermined coefficients

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the form of the particular solution?

    (A) $y_p(t) = Ae^{2t}$

    (B) $y_p(t) = Ae^{-2t}$

    ☆ (C) $y_p(t) = Ae^t$

    (D) $y_p(t) = Ate^t$

    (E) Don't know

Method of undetermined coefficients

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
    - What is the value of A that gives the particular solution (Ae^t) ?

(A) A = 1

(B) A = 3

(C) A = 1/3

☆ (D) A = -1/3

(E) Don't know.

Method of undetermined coefficients

- Example 6. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the solution to the associated homogeneous equation?

★ (A) $y_h(t) = C_1 e^{2t} + C_2 e^{-2t}$

(B) $y_h(t) = C_1 \cos(2t) + C$ …

(C) $y_h(t) = C$ …

… $C_1 e^{2t} + C_2 e^{-2t} + e^t$

(E) Don't know.

Same as the last example

Method of undetermined coefficients

- Example 6. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t}$

$$(Ae^{2t})'' - 4Ae^{2t} = 0 \ !$$

(B) $y_p(t) = Ae^{-2t}$

☆ (C) $y_p(t) = Ate^{2t}$

(D) $y_p(t) = Ae^{t}$

(E) $y_p(t) = Ate^{t}$

- Simpler example in which the RHS is a solution to the homogeneous problem.

$$y' - y = e^t$$

$$e^{-t}y' - e^{-t}y = 1$$

$$y = te^t + Ce^t$$

- General rule: when your guess at y_p makes LHS=0, try multiplying it by t.

Method of undetermined coefficients

- Example 6. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the value of A that gives the particular solution (Ate^{2t})?

(A) A = 1

(B) A = 4

(C) A = -4

☆ (D) A = 1/4

(E) A = -1/4

$$(Ate^{2t})' = Ae^{2t} + 2Ate^{2t}$$

$$(Ate^{2t})'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$$

$$= 4Ae^{2t} + 4Ate^{2t}$$

$$\left(Ate^{2t}\right)'' - 4\left(Ate^{2t}\right) = 4Ae^{2t}$$

Need: $= e^{2t}$

Method of undetermined coefficients

- Example 7. Find the general solution to $y'' - 4y = \cos(2t)$.
  - What is the form of the particular solution?

☆ (A) $y_p(t) = A\cos(2t)$

(B) $y_p(t) = A\sin(2t)$

☆ (C) $y_p(t) = A\cos(2t) + B\sin(2t)$

(D) $y_p(t) = t(A\cos(2t) + B\sin(2t))$

(E) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

Challenge: What small change to the DE makes (D) correct?

Method of undetermined coefficients

- Example 7. Find the general solution to $y'' + y' - 4y = \cos(2t)$.
    - What is the form of the particular solution?

(A) $y_p(t) = A\cos(2t)$

(B) $y_p(t) = A\sin(2t)$

★ (C) $y_p(t) = A\cos(2t) + B\sin(2t)$

(D) $y_p(t) = t(A\cos(2t) + B\sin(2t))$

(E) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

Method of undetermined coefficients

- Example 8. Find the general solution to $y'' - 4y = t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = At^3$

(B) $y_p(t) = At^3 + Bt^2 + Ct$

★ (C) $y_p(t) = At^3 + Bt^2 + Ct + D$

(D) $y_p(t) = At^3 + Be^{2t} + Ce^{-2t}$ ← waste of time including solution to homogeneous eq.

(E) Don't know.

Method of undetermined coefficients

- When RHS is sum of terms:

$$y'' - 4y = \cos(2t) + t^3$$

$$y_p(t) = A\cos(2t) + B\sin(2t) + Ct^3 + Dt^2 + Et + F$$

or

$$y_{p_1}(t) = A\cos(2t) + B\sin(2t)$$

$$y_{p_2}(t) = Ct^3 + Dt^2 + Et + F$$

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$$