Today

- Systems with complex eigenvalues - how to figure out rotation
- Systems with a repeated eigenvalue
- Summary of 2x2 systems with constant coefficients.

Direction of rotation in complex eigenvalue case

$$x' = x - 8y$$
$$y' = 8x + y$$

(A) Solutions rotate clockwise and decay exponentially.

(B) Solutions grow exponentially without oscillating.

(C) Solutions rotate clockwise and grow exponentially.

☆(D) Solutions rotate counterclockwise and grow exponentially.

Direction of rotation in complex eigenvalue case

$$x' = x - 8y$$
$$y' = 8x + y$$

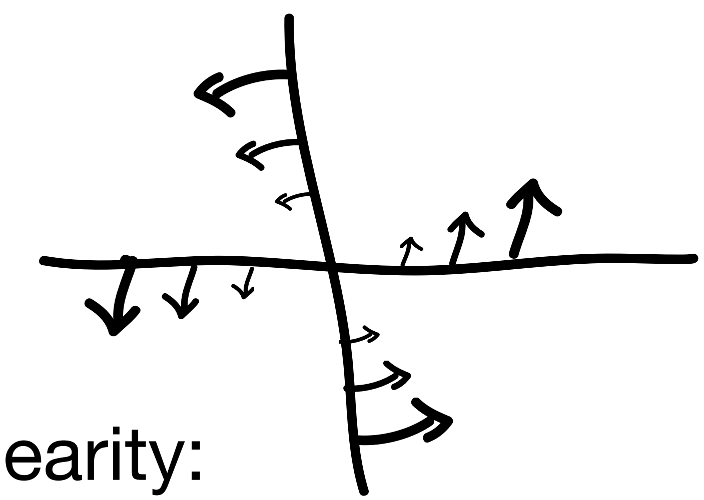
$$\bar{x}' = \begin{pmatrix} 1 & -8 \\ 8 & 1 \end{pmatrix} \bar{x}$$

$$\lambda^2 - \operatorname{tr} A \lambda + \det A = 0$$

$$\lambda^2 - 2\lambda + 65 = 0$$

$$\lambda = 1 \pm i8$$

Exponential growth



(by linearity:
A(-x)=-Ax)

$$\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \bar{x}' = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \bar{x}' = \begin{pmatrix} -8 \\ 1 \end{pmatrix}$$

Counterclockwise rotation!

Repeated eigenvalues

- What happens when you get two identical eigenvalues?
- Two cases:

1. The single eigenvalue has two distinct eigenvectors.
2. There is only one eigenvector (matrix is defective).

1. $\overline{\mathbf{x}}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \overline{\mathbf{x}}$

2. $\overline{\mathbf{x}}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \overline{\mathbf{x}}$

Repeated eigenvalues

1. $\overline{\mathbf{x}}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \overline{\mathbf{x}}$

$$\det(A - \lambda I) = (\lambda - 3)^2 = 0$$

$$\lambda = 3$$

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v} = 0$$

All vectors solve this so choose any two independent vectors:

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2. $\overline{\mathbf{x}}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \overline{\mathbf{x}}$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = 2$$

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{v} = 0$$

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

<-- only 1 evector!

$$\mathbf{x}(t) = C_1 e^{2t} \mathbf{v} + C_2 e^{2t} (\mathbf{w} + t\mathbf{v})$$

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

$$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

<-- called "generalized evector"

Systems of ODEs - steps for solving (2x2)

- Find evalues (λ) and evectors (**v**) or generalized evectors (**w**) of A:
  - Distinct real - $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2}$

    where λ and $\mathbf{v_i}$ solve (A - λI) $\mathbf{v_i}$ =0.
  - Complex - $\mathbf{x}(\mathbf{t}) = e^{\alpha t}\left[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + C_2\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)\right]$

    where $\lambda_1 = \alpha + \beta i$ and $\mathbf{v_1} = \mathbf{a} + \mathbf{b}i$.
  - Repeated with two eigenvectors (diagonal matrices only) -

    $$\mathbf{x}(t) = C_1 e^{\lambda t}\mathbf{v_1} + C_2 e^{\lambda t}\mathbf{v_2}$$
  - Repeated with one eigenvector - $\mathbf{x}(t) = C_1 e^{\lambda t}\mathbf{v} + C_2 e^{\lambda t}(\mathbf{w} + t\mathbf{v})$

    where λ and **v** solve (A - λI) **v** = **0** and **w** solves (A - λI) **w** = **v**.

Steady state - two notions

- Forced mass-spring systems - long term behaviour after transient dies down.
  - If the IC isn't right on $y_p(t)$, the homog solution decays exponentially (for $\alpha<0$) so eventually only y_p remains.

$$y(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t)) + y_p(t)$$

  - SS can be oscillation (not constant).
- Constant solutions of a system of ODEs (discussed in the next slides).
  - Transient may decay or grow exponentially.
  - Always constant solutions!

Summary - homogeneous 2x2 systems

Steady states - constant solutions (set x'=0 and solve Ax=0).

- For the system of equations $\mathbf{x}' = A\mathbf{x}$, we always have $\mathbf{x}(t) = \mathbf{0}$ as a steady state solution.
- If *A* is a singular matrix with $A\mathbf{v} = \mathbf{0}$ then $\mathbf{x}(t) = \mathbf{v}$ is also a steady state solution.
    - In fact, $\mathbf{x}(t) = c\mathbf{v}$ is a steady state for all *c.*
    - It is also an eigenvector associated with eigenvalue $\lambda = 0$.
- If *A* is nonsingular then $\mathbf{x}(t) = \mathbf{0}$ is the only steady state.

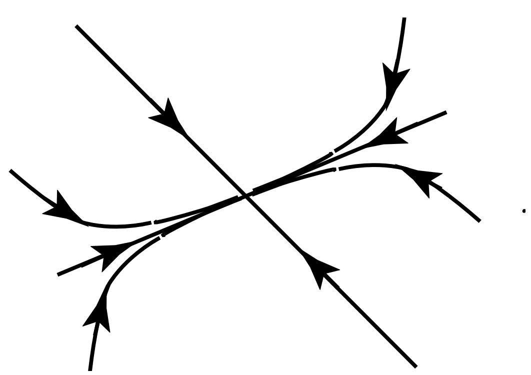
Summary - homogeneous 2x2 systems

Steady states

- Steady states are classified by the nature of the surrounding solutions:

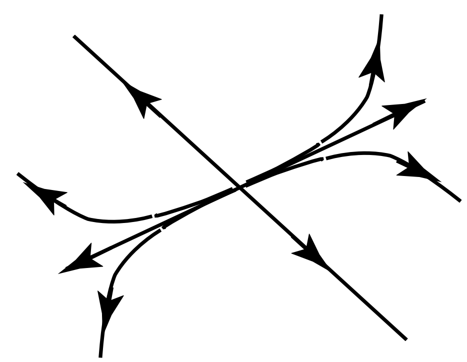
stable node

- real negative evalues



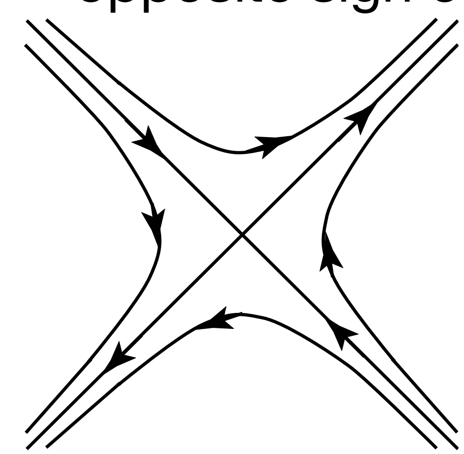
unstable node

- real positive evalues



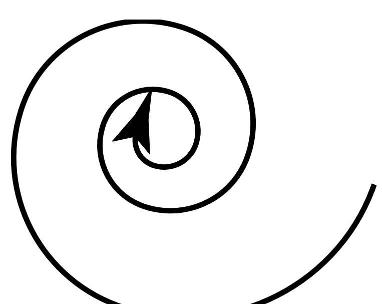
saddle

- opposite sign evalues



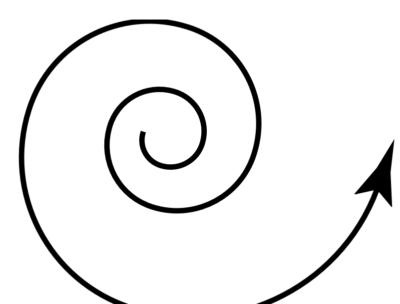
stable spiral

- complex evalues, negative real part



unstable spiral

- complex evalues, positive real part



Summary - homogeneous 2x2 systems

- Quick way to determine how all other solutions behave:

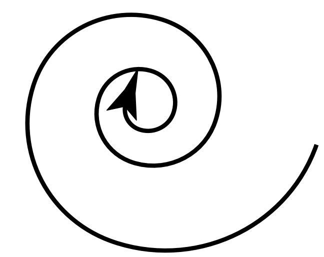
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc \\ &= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) \qquad = 0 \end{aligned}$$

Summary - homogeneous 2x2 systems

- When do the solutions spiral IN to the origin?

$$\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$$



ensures negative real part

★ (A) $\begin{cases} \mathrm{tr}A < 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

$$\lambda = \boxed{\frac{\mathrm{tr}A}{2}} \pm \boxed{\frac{\sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}}$$

ensures complex evalue

(B) $\begin{cases} \mathrm{tr}A > 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

(C) $\begin{cases} \mathrm{tr}A < 0, \ \det(A) > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

(D) $\begin{cases} \mathrm{tr}A > 0, \ \det(A) > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

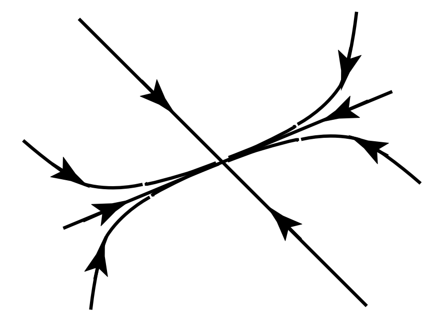
(E) Explain, please.

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

Summary - homogeneous 2x2 systems

- When is the origin a stable node?

$$\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$$



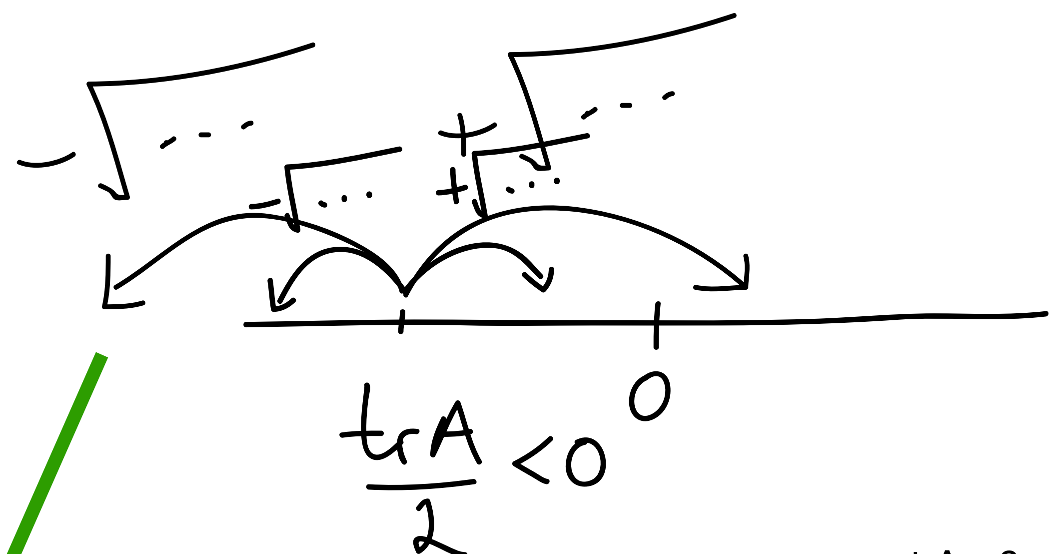
(A) $\begin{cases} \mathrm{tr}A < 0 \\ ~~(\mathrm{tr}A)^2 < 4\det A~~ \end{cases}$

not complex!

(B) $\begin{cases} \mathrm{tr}A > 0 \\ ~~(\mathrm{tr}A)^2 < 4\det A~~ \end{cases}$

★(C) $\begin{cases} \mathrm{tr}A < 0, \ \det(A) > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

(D) $\begin{cases} \mathrm{tr}A < 0, \ \det(A) < 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$



trA=-6, detA=-1
trA=-6, detA=2

(E) Explain, please.

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

Using the trace/determinant plane to classify systems

- Classify the steady state of the equation x'=Ax.

$$A = \begin{pmatrix} 1 & 1 \\ -6 & -4 \end{pmatrix}$$

$\mathrm{tr}(A) =$ -3 so ~~some~~ all solutions decay.

$\det(A) =$ 2 > 0 so not a saddle.

$(\mathrm{tr}A)^2 - 4\det(A) =$ 1 > 0 so not complex e-values.

Therefore, two negative e-values => stable node.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

$\mathrm{tr}(A) =$ 4 so ~~some~~ all solutions grow.

$\det(A) =$ 3 > 0 so not a saddle.

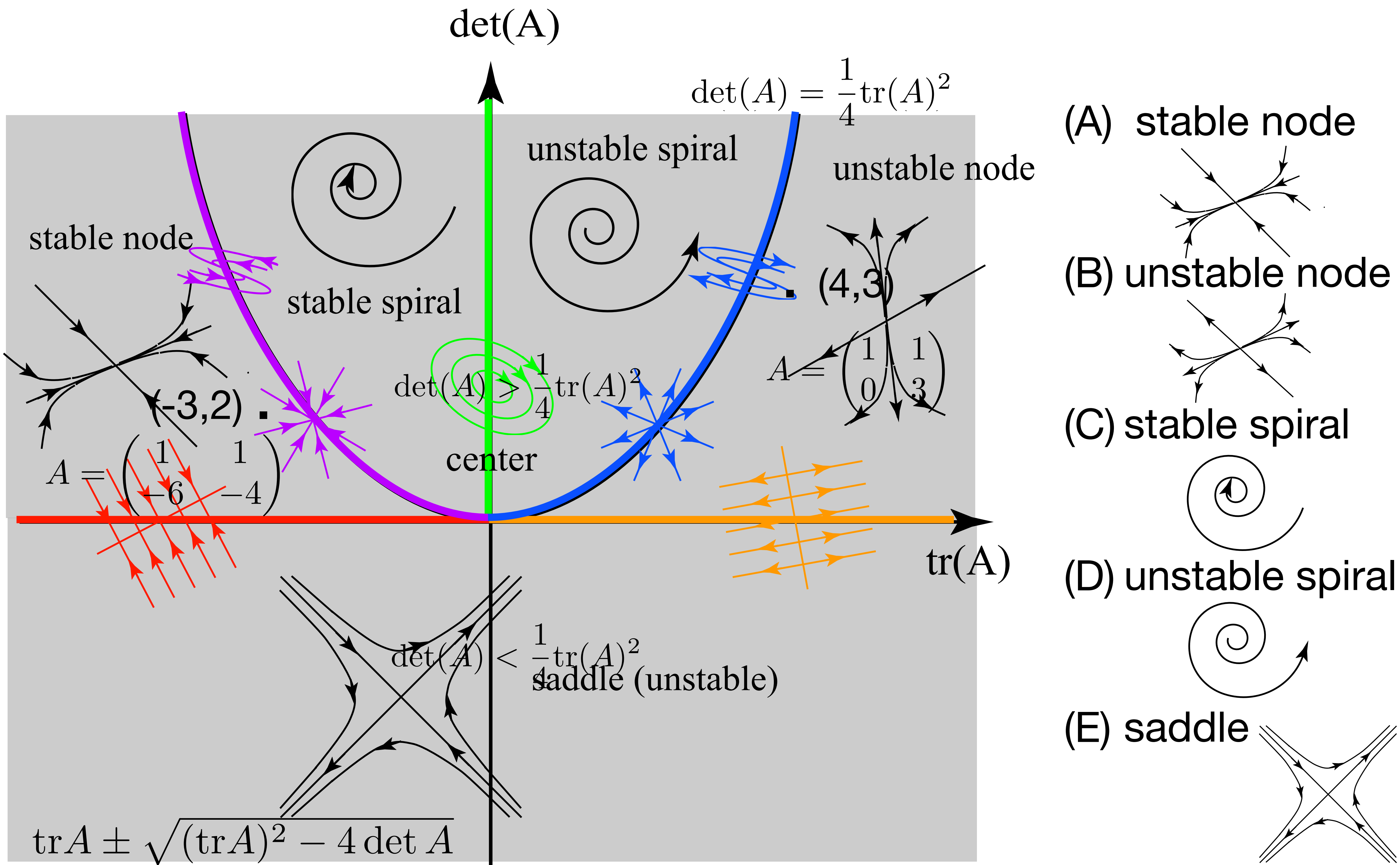
$(\mathrm{tr}A)^2 - 4\det(A) =$ 4 > 0 so not complex e-values.

Therefore, two positive e-values => unstable node.

When given numbers, just find e-values but with parameters, need a way to derive conditions.

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

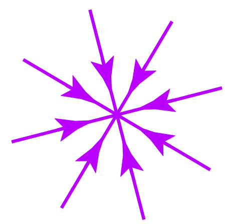
Summary - homogeneous 2x2 systems



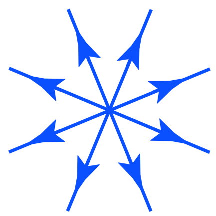
$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

Summary - homogeneous 2x2 systems

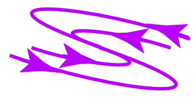
Repeated evalue cases:



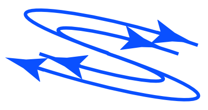
$\lambda<0$, two indep. evectors.



$\lambda>0$, two indep. evectors.

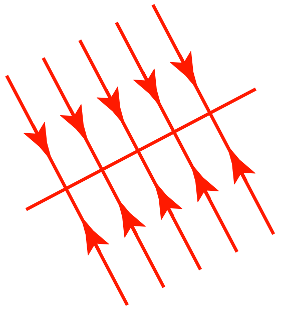


$\lambda<0$, only one evector.

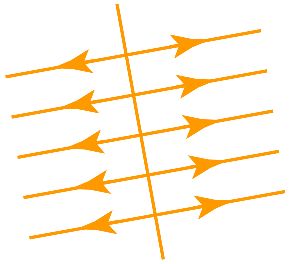


$\lambda>0$, only one evector.

One zero evalue (singular matrix):



$\lambda_1=0$, $\lambda_2<0$,



$\lambda_1=0$, $\lambda_2>0$,