

Today

- Summary of steps for solving the Diffusion Equation with homogeneous Dirichlet or Neumann BCs using Fourier Series.
- Nonhomogeneous BCs
- Mixed Dirichlet/Neumann BCs
- Method of Undetermined Coefficients and Fourier Series

Using Fourier Series to solve the Diffusion Equation

- Steps to solving the PDE:
 - Determine the eigenfunctions for the problem (look at BCs).
 - Represent the IC $u(x,0)=f(x)$ by a sum of eigenfunctions (Fourier series).
 - Write down the solution by inserting $e^{\lambda t}$ into each term of the FS.

$u_t = Du_{xx}$ \longrightarrow PDE determines all possible eigenfunctions.

$\left. \frac{du}{dx} \right|_{x=0,L} = 0$ \longrightarrow BCs select a subset of the eigenfunctions.

$u(x, 0) = f(x)$ \longrightarrow IC is satisfied by adding up eigenfunctions.

Using Fourier Series to solve the Diffusion Equation

$u_t = D u_{xx}$  PDE determines all possible eigenfunctions.

Let's look for all possible eigenfunctions:

$$D v_{xx}(x) = \lambda v(x)$$

Case I: $\lambda < 0$. $v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$ and $w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$

For each value of $\lambda < 0$, these are both eigenfunctions.

Case II: $\lambda = 0$. $v_{xx} = 0 \Rightarrow v_x = C_1 \Rightarrow v(x) = C_1 x + C_2$

The eigenfunctions are therefore $v(x) = 1$ and $v(x) = x$.

Case III: $\lambda > 0$. $v_\lambda(x) = e^{\sqrt{\frac{\lambda}{D}}x}$ and $w_\lambda(x) = e^{-\sqrt{\frac{\lambda}{D}}x}$

These won't come up so I'll drop Case III.

Using Fourier Series to solve the Diffusion Equation

$u_t = Du_{xx}$ \longrightarrow PDE determines all possible eigenfunctions.

$\left. \frac{du}{dx} \right|_{x=0,L} = 0$ \longrightarrow BCs select a subset of the eigenfunctions.

Case I: $\lambda < 0$. $v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$ and ~~$w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$~~

The BC at $x=0$ only works for $v_\lambda(x)$ and the BC at $x=L$ only works for certain λ , in particular $\lambda = -n^2\pi^2 D/L^2$.

Case II: $\lambda=0$. $v(x) = 1$ and ~~$v(x) = x$~~

Represent IC $u(x,0) = f(x)$ by $u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos \frac{n\pi x}{L}$$

\uparrow
 e^{0t}

Using Fourier Series to solve the Diffusion Equation

$u_t = D u_{xx}$ \longrightarrow PDE determines all possible eigenfunctions.

$u(0, t) = u(2, t) = 0$ \longrightarrow BCs select a subset of the eigenfunctions.

Case I: $\lambda < 0$. ~~$v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$~~ and $w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$

The BC at $x=0$ only works for $w_\lambda(x)$ and the BC at $x=L$ only works for certain λ , in particular $\lambda = -n^2\pi^2 D/L^2$.

Case II: $\lambda=0$. ~~$v(x) = 1$~~ and ~~$v(x) = x$~~

Represent IC $u(x,0) = f(x)$ by $u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin \frac{n\pi x}{L}$$

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$\left. \frac{du}{dx} \right|_{x=0,2} = 0$$

$$u(x, 0) = \sin \frac{3\pi x}{2}$$

(A) $u(x, t) = e^{-9\pi^2 t} \cos \frac{3\pi x}{2}$  doesn't satisfy IC.

(B) $u(x, t) = e^{-9\pi^2 t} \sin \frac{3\pi x}{2}$  don't satisfy BCs.

(C) $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin \frac{n\pi x}{2}$  $b_n = \int_0^2 \sin \frac{3\pi x}{2} \sin \frac{n\pi x}{2} dx$

★ (D) $u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t} \cos \frac{n\pi x}{2}$ $a_n = \int_0^2 \sin \frac{3\pi x}{2} \cos \frac{n\pi x}{2} dx$

...with nonhomogeneous boundary conditions

$$u_t = Du_{xx}$$

$$u(0, t) = 0$$

$$u(2, t) = 4$$

→ Nonhomogeneous BCs

an eigenfunction for the homogeneous BCs

Case I: $\lambda < 0$. ~~$v_\lambda(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$~~ and $w_\lambda(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$

The BC at $x=0$ only works for $w_\lambda(x)$ and the BC at $x=L$ almost works for certain λ , in particular $\lambda = -n^2\pi^2 D/L^2$.

Case II: $\lambda=0$. ~~$v(x) = 1$~~ and $v(x) = 2x$ ← a particular eigenfunction for the inhomogeneous BCs

Ultimately, we want $u(x, t) = 2x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin \frac{n\pi x}{L}$

What function do we use to calculate the Fourier series $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$?

- (A) $u(x, 0)$ (B) $u(x, 0) - 2$ (C) $u(x, 0) - 2x$ (D) $u(x, 0) + 2x$



...with nonhomogeneous boundary conditions

- Solve the Diffusion Equation with nonhomogeneous BCs:

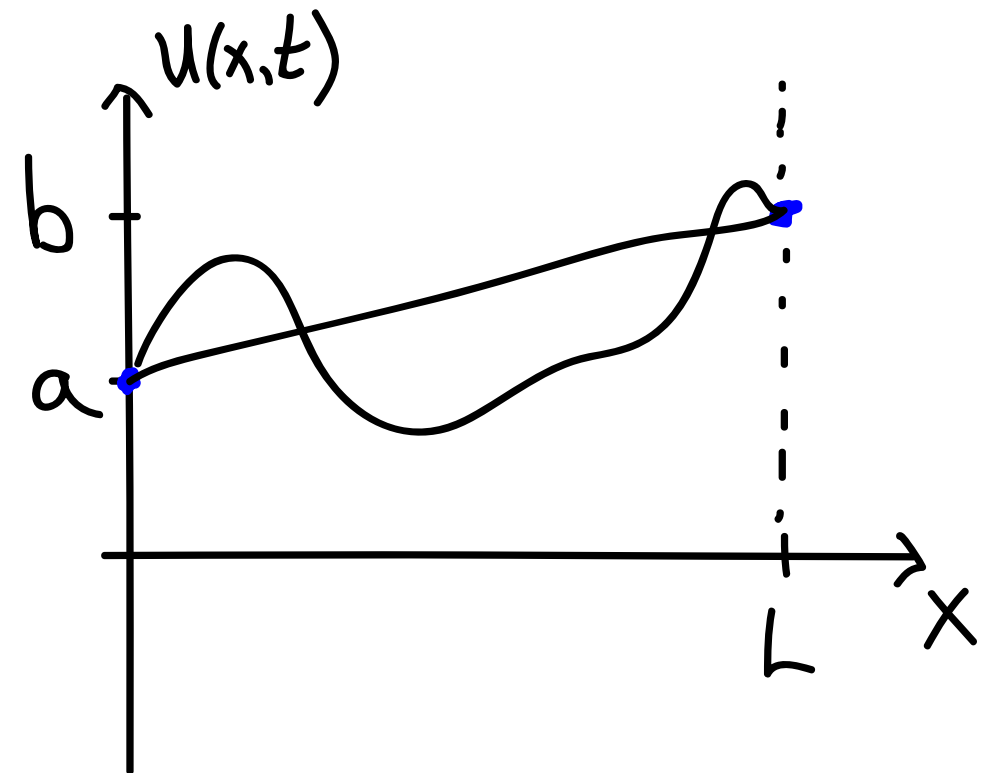
$$u_t = Du_{xx}$$

$$u(0, t) = a$$

$$u(L, t) = b$$

$$u(x, 0) = f(x)$$

- Recall - rate of change is proportional to curvature so bumps get ironed out.



$$v(x, t) = u(x, t) - \left(a + \frac{b-a}{L}x \right)$$

$$\left. \begin{array}{l} v_t = u_t \\ v_{xx} = u_{xx} \end{array} \right\} \Rightarrow v_t = Dv_{xx}$$

$$v(0, t) = u(0, t) - a = 0$$

$$v(L, t) = u(L, t) - b = 0$$

$$v(x, 0) = u(x, 0) - \left(a + \frac{b-a}{L}x \right)$$

- $v(x, t)$ satisfies the Diffusion Eq with homogeneous Dirichlet BCs and a new IC.

...with nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$

$$u(0, t) = 9$$

$$u(2, t) = 5$$

$$u(x, 0) = \sin \frac{3\pi x}{2}$$

$$(A) \quad u(x, t) = e^{-9\pi^2 t} \sin \frac{3\pi x}{2}$$

$$(B) \quad u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin \frac{n\pi x}{2}$$

$$\star (C) \quad u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin \frac{n\pi x}{2} + 9 - 2x$$

$$(D) \quad u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 t} \cos \frac{n\pi x}{2}$$

$$\text{where } b_n = \int_0^2 \left(\sin \frac{3\pi x}{2} - 9 + 2x \right) \sin \frac{n\pi x}{2} dx$$

...with nonhomogeneous boundary conditions

- How would you solve this one?

$$u_t = 4u_{xx}$$

$$\left. \frac{du}{dx} \right|_{x=0,2} = -2$$

$$u(x, 0) = \cos \frac{3\pi x}{2}$$

For you to think about...

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

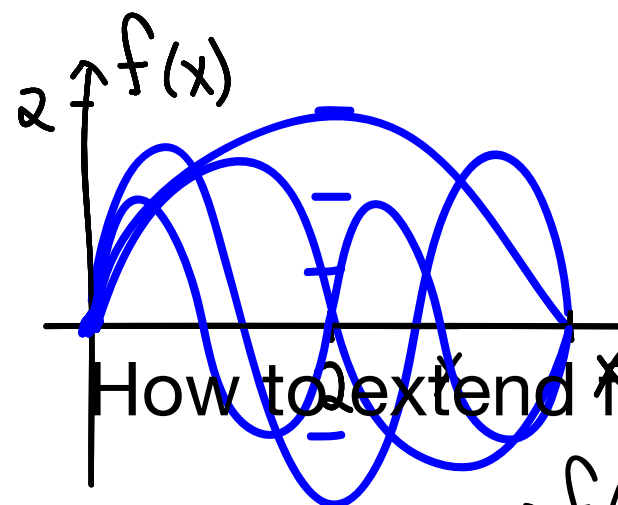
Use sines? cosines?

$$u(0, t) = 0 \quad \left. \frac{du}{dx} \right|_{x=2} = 0$$

Should be zero at $x=0$ so definitely sine functions.

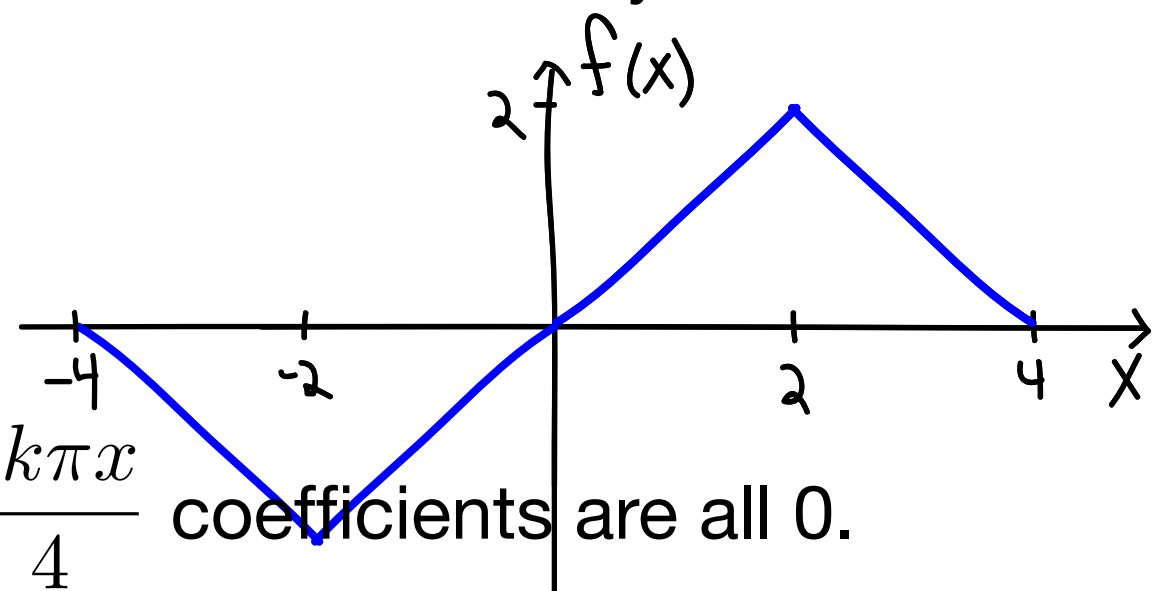
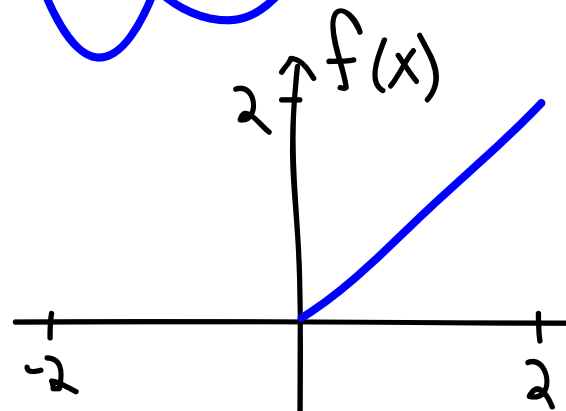
$$u(x, 0) = x$$

Zero slope at $x=2$ so extend to $x=4$ and choose periods to get the slope right.



$$\sin \frac{n\pi x}{4}: \quad \sin \frac{\pi x}{4} \quad \cancel{\sin \frac{2\pi x}{4}} \quad \sin \frac{3\pi x}{4} \quad \cancel{\sin \frac{4\pi x}{4}}$$

How to extend $f(x)$ so that its Fourier sine series has only odd values of n ?



Extension is “even” about $x=2$ so $\sin \frac{2k\pi x}{4}$ coefficients are all 0.

Using Fourier Series with Method of Undet. Coeff.

- Find the solution to the following problem:

$$y'' + 16y = 4 \sin(\pi t) - 3 \sin(2\pi t)$$

$$y(t) = y_h(t) + y_p(t) \quad \text{where}$$

★ (A) $y_p(t) = A \sin(\pi t) + B \sin(2\pi t)$ Because no y' term, the cosine coefficients are all 0.

★ (B) $y_p(t) = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$ Technically ok but all B_n for $n > 2$ will be zero. Need whole family so include cosines.

★ (C) $y_p(t) = A \sin(\pi t) + B \cos(\pi t) + C \sin(2\pi t) + D \cos(2\pi t)$

★ (D) $y_p(t) = \sum_{n=1}^{\infty} A_n \cos(n\pi t) + \sum_{n=1}^{\infty} B_n \sin(n\pi t)$ Technically ok but all B_n for $n > 2$ and all A_n will be zero.

- Note: we definitely did not use 4 and -3 as our coefficients for the guess!

Using Fourier Series with Method of Undet. Coeff.

- Find the solution to the following problem:

$$y'' + 16y = \sum_{n=1}^8 b_n \sin(n\pi t) \quad \text{where the } b_n \text{ are given values.}$$

When the RHS is a sum, we can work with one term at a time so let's just focus on one of them, but not specify which:

$$y'' + 16y = b_n \sin(n\pi t)$$

Because there is no y' term, we can include only the sine function in our guess:

$$y_p(t) = B_n \sin(n\pi t)$$

$$y_p''(t) + 16y_p(t) = -n^2\pi^2 B_n \sin(n\pi t) + 16B_n \sin(n\pi t) = b_n \sin(n\pi t)$$

$$B_n = \frac{b_n}{16 - n^2\pi^2} \quad y_p(t) = \sum_{n=1}^8 B_n \sin(n\pi t) \quad \text{What if the 16 had been } 4\pi^2?$$