Today

- Solving ODEs using Laplace transforms
- The Heaviside and associated step and ramp functions
- ODE with a ramped forcing function

Solving IVPs using Laplace transforms - complex

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+13}$$

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$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

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Solving IVPs using Laplace transforms - real

- Solve the equation $y'' + 6y' + 5y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2 + 6s + 5}$$

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Solving IVPs using Laplace transforms - nonhomog

- What is the transformed equation for the IVP

$$y' + 6y = e^{2t}$$
$$y(0) = 2$$

(A) $Y'(s) + 6Y(s) = \dfrac{1}{s+2}$

(B) $Y'(s) + 6Y(s) = \dfrac{1}{s-2}$

(C) $sY(s) + 2 + 6Y(s) = \dfrac{1}{s+2}$

(D) $sY(s) - 2 + 6Y(s) = \dfrac{1}{s-2}$

(E) Explain, please.

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{(2-s)t}\ dt$$
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$$= \frac{2}{s+6} + \frac{1}{(s-2)(s+6)}$$

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$$y(t) = 2e^{-6t} + \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+6)}\right)$$

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$$1 = A(s+6) + B(s-2)$$

$$(s=2) \quad 1 = 8A$$

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$$y(t) = \frac{15}{8}e^{-6t} + \frac{1}{8}e^{2t}$$

$$y_h(t) = Ce^{-6t}$$

$$C = \frac{15}{8} \qquad y_p(t) = \frac{1}{8}e^{2t}$$

Solving IVPs using Laplace transforms

- With a forcing term, the equation

$$ay'' + by' + cy = g(t)$$

has Laplace transform

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- Solving for Y(s):

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

transform of homogeneous solution with two degrees of freedom (y(0) and y'(0) act like C_1 and C_2.

transform of particular solution

Solving IVPs using Laplace transforms

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$$Y_h(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2} \quad \rightarrow \quad y_h(t) = Ae^{r_1 t} + Be^{r_2 t}$$

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$$Y_h(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2}$$

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$$Y_h(s) = \frac{A}{s - r} + \frac{B}{(s - r)^2}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

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$$Y_h(s) = \frac{A}{s-r} + \frac{B}{(s-r)^2} \quad \longrightarrow \quad y_h(t) = Ae^{rt} + Bte^{rt}$$

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Solving IVPs using Laplace transforms

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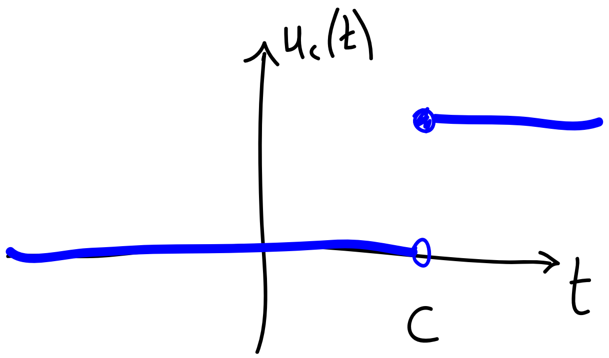
Laplace transforms (so far)

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

Step function forcing

- We define the Heaviside function $u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$



- In WW, $u_c(t) = u(t-c) = h(t-a)$

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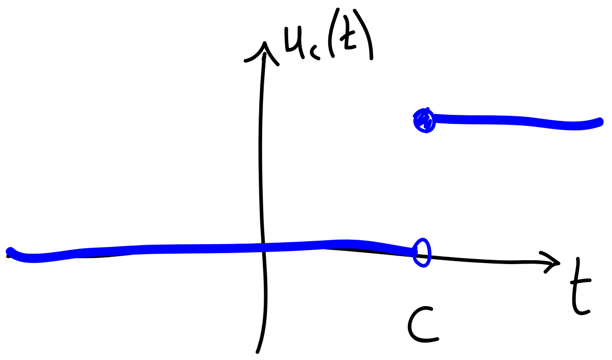

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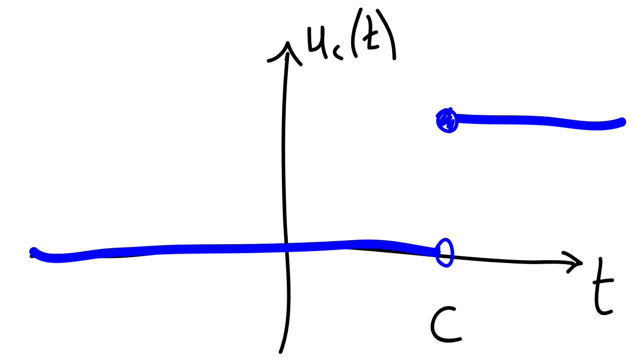

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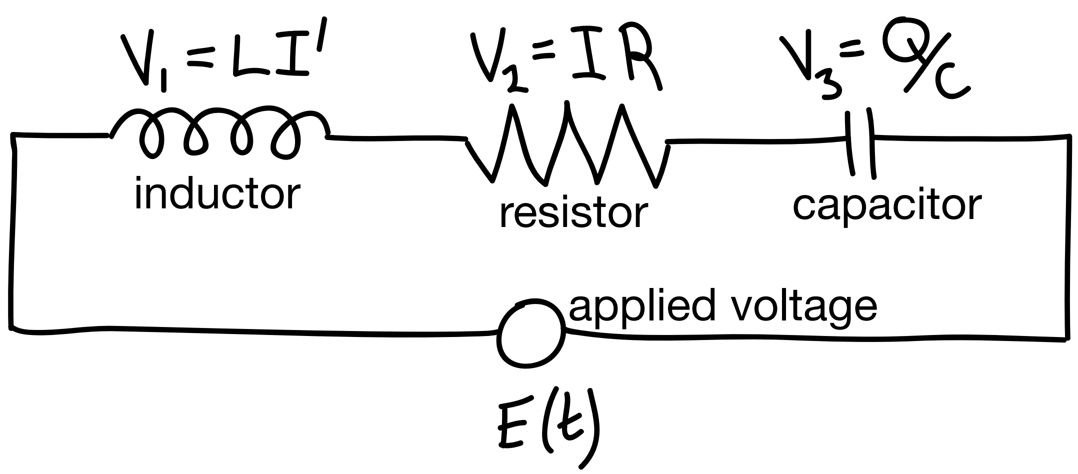

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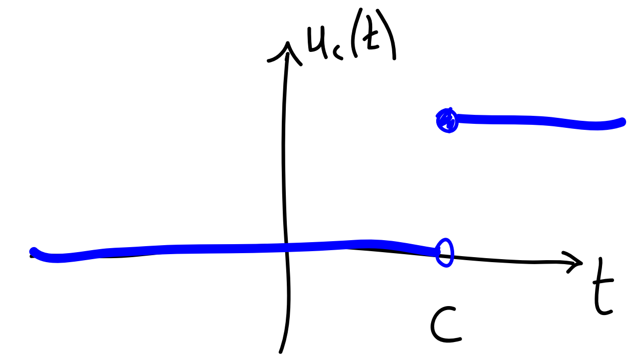

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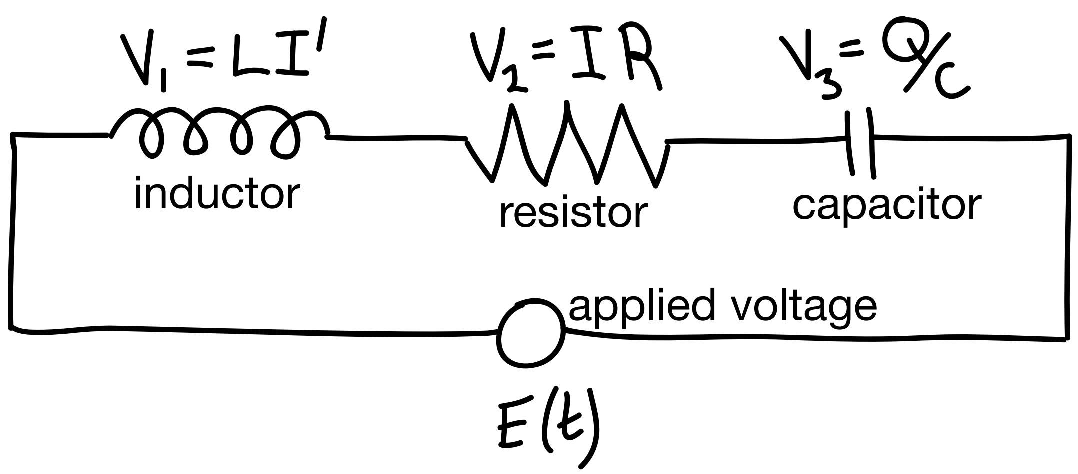

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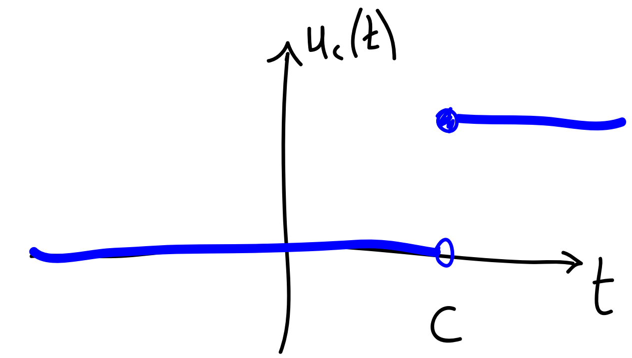

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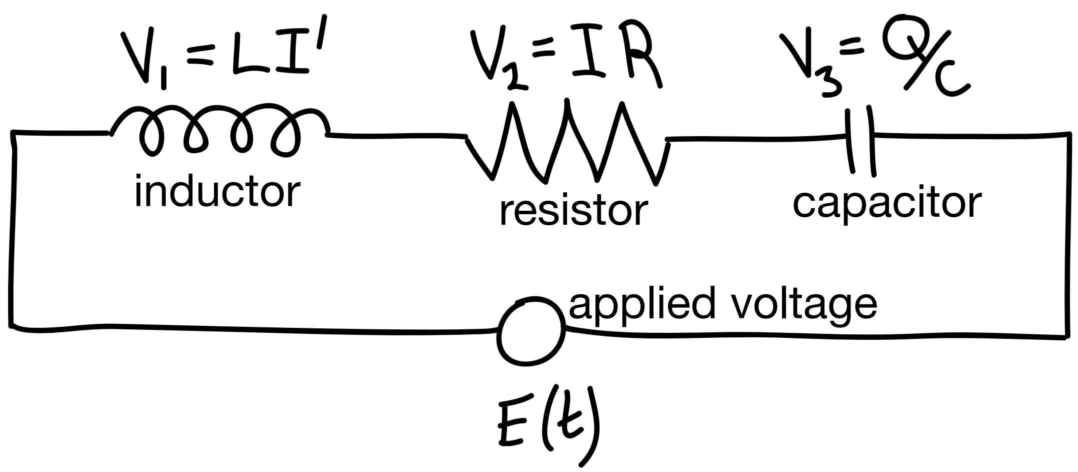

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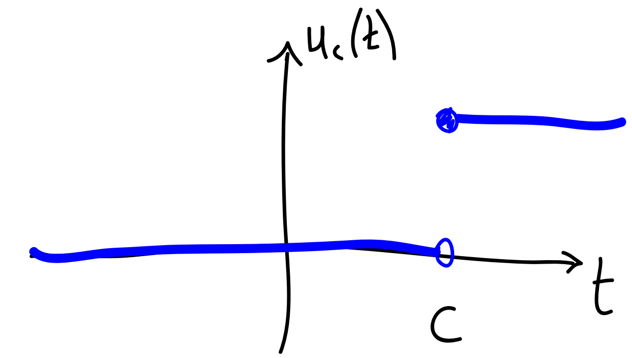

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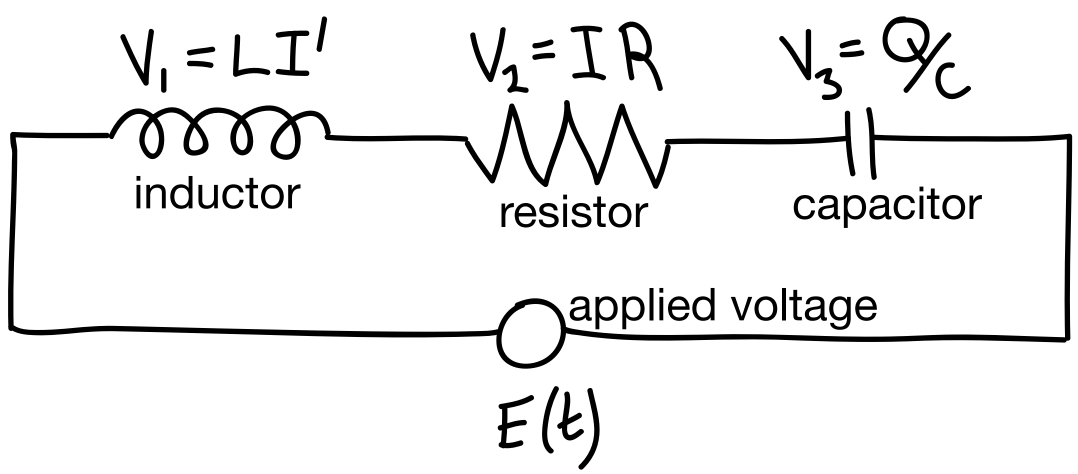

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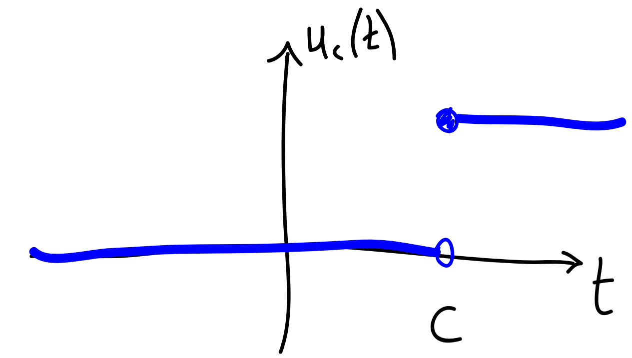

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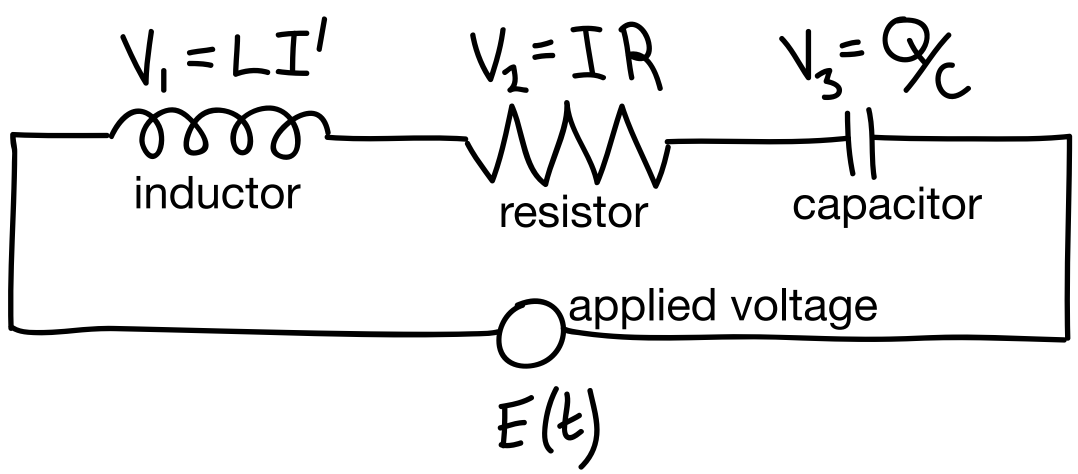

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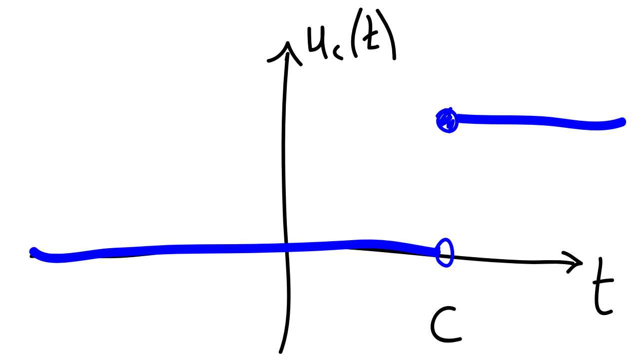

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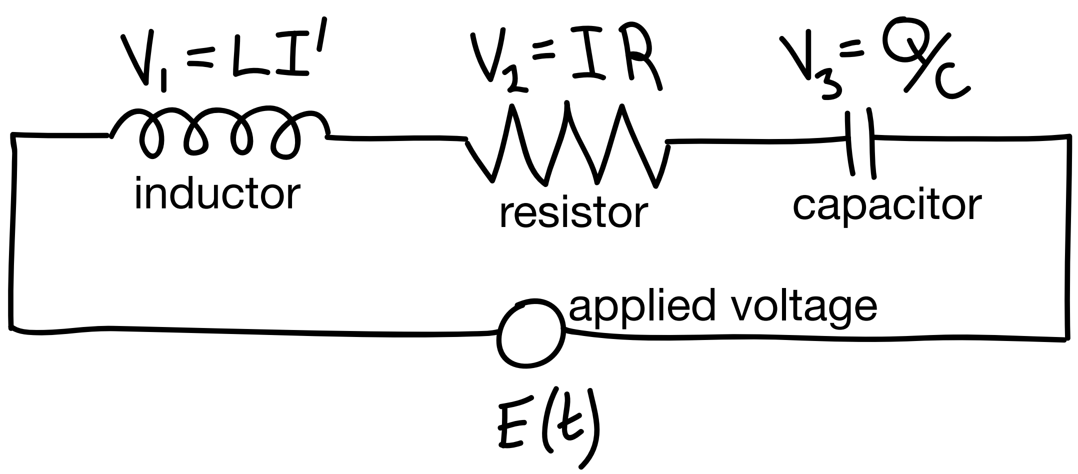

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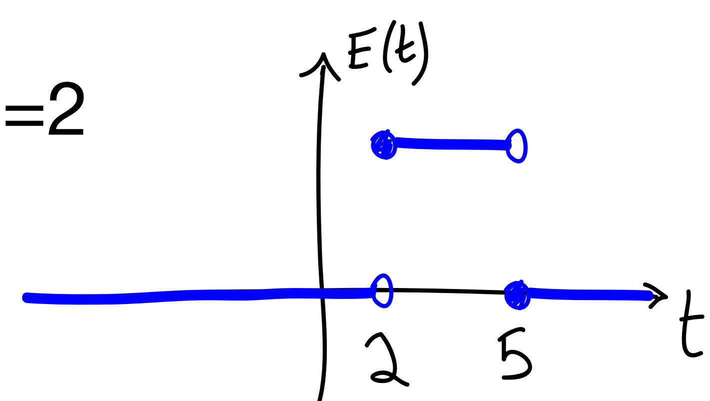
- For example, turn E on at t=2 and off again at t=5:



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- Use the Heaviside function to rewrite $g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$

(A) $g(t) = u_2(t) + u_5(t)$

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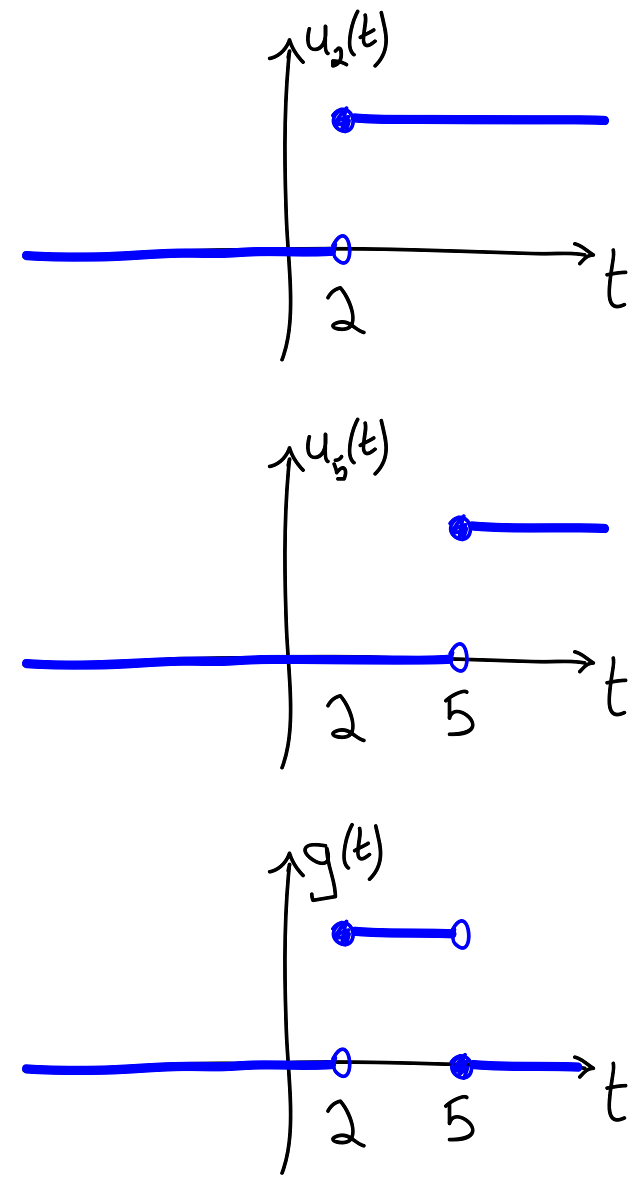

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messier with transforms

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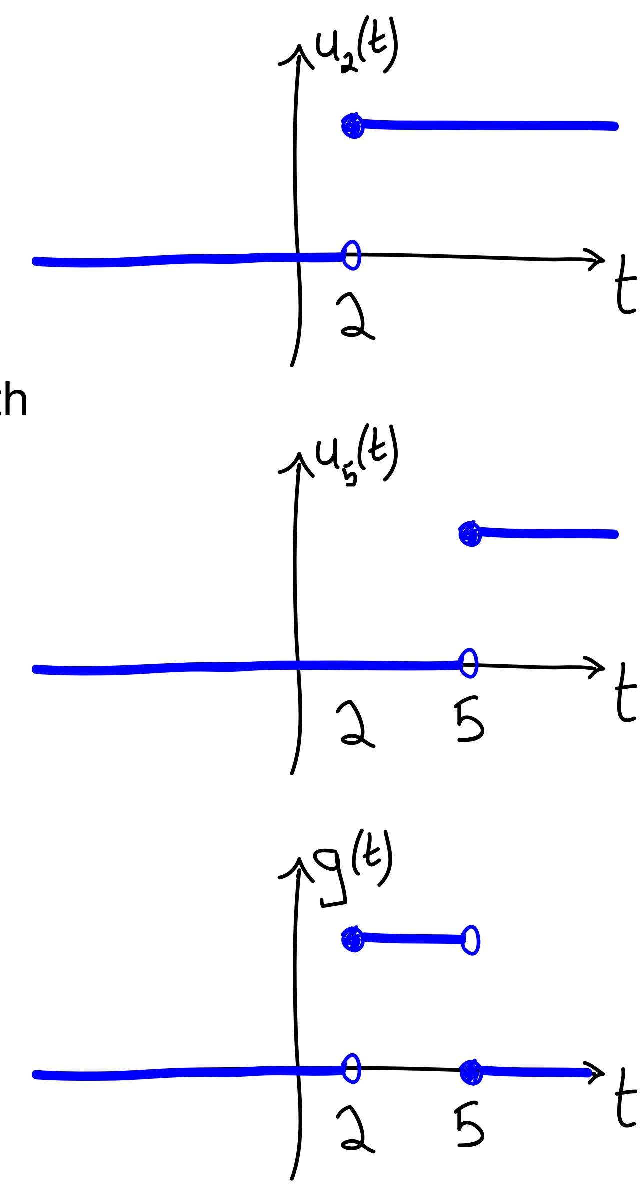

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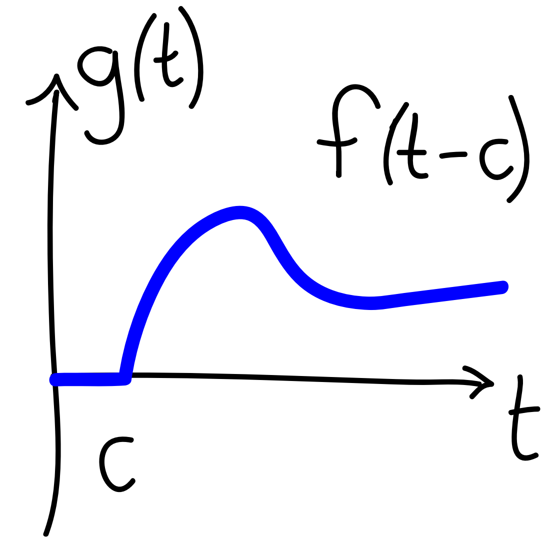

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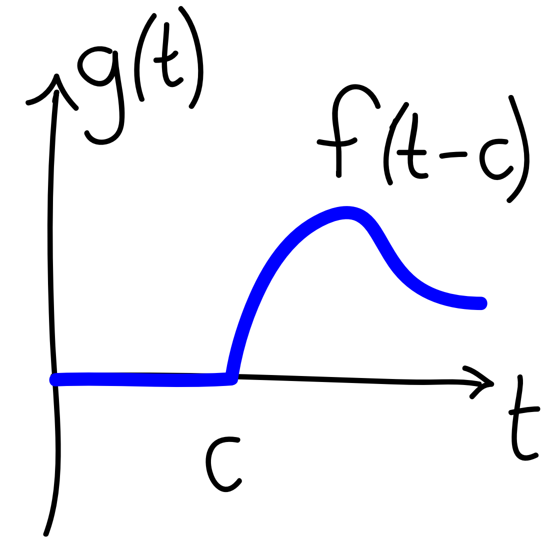

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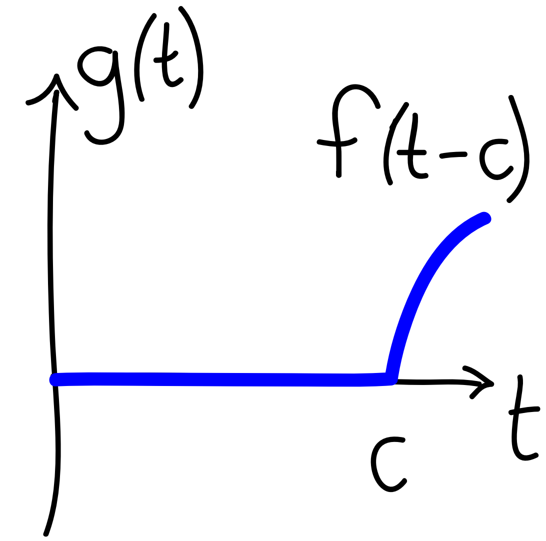

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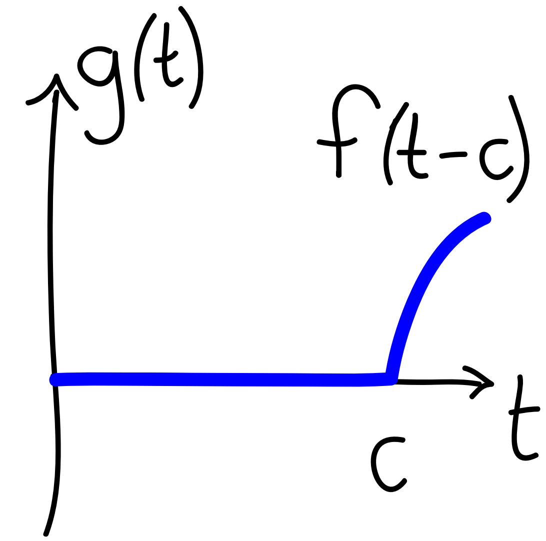

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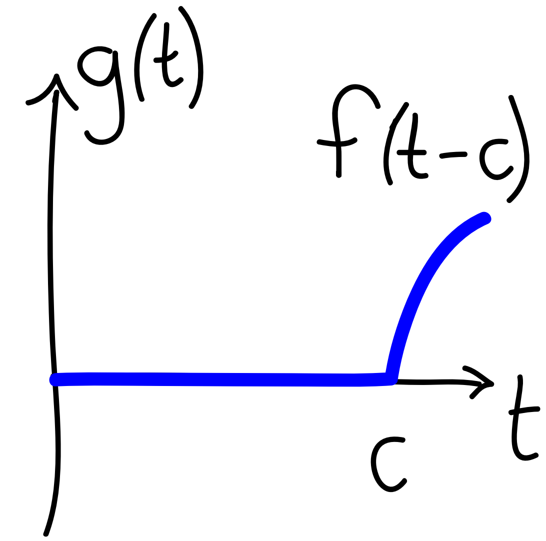

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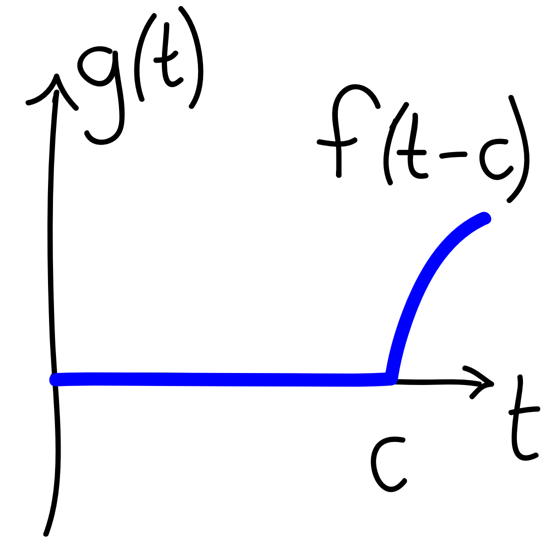

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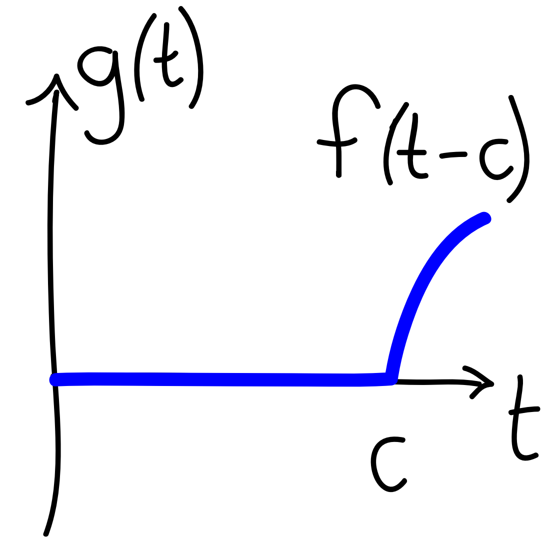

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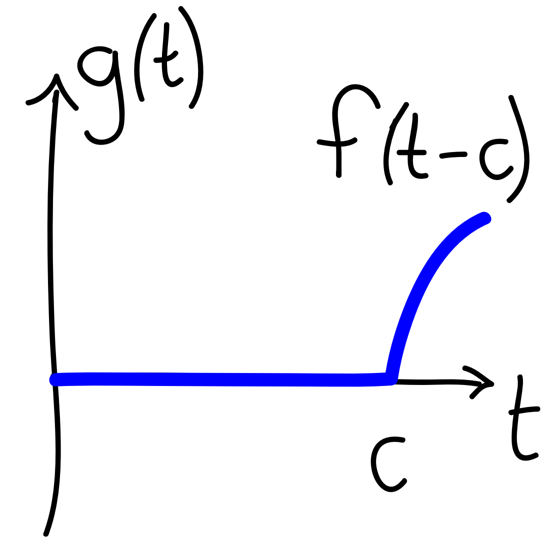

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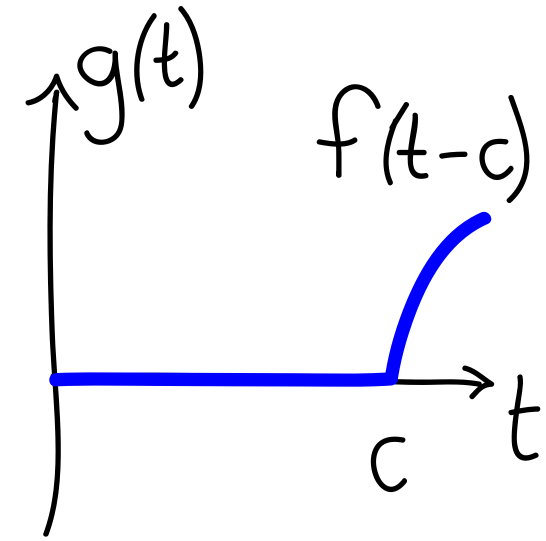

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- Solve using Laplace transforms:

$$y'' + 2y' + 10y = g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

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- So we just need h(t) and we're done.

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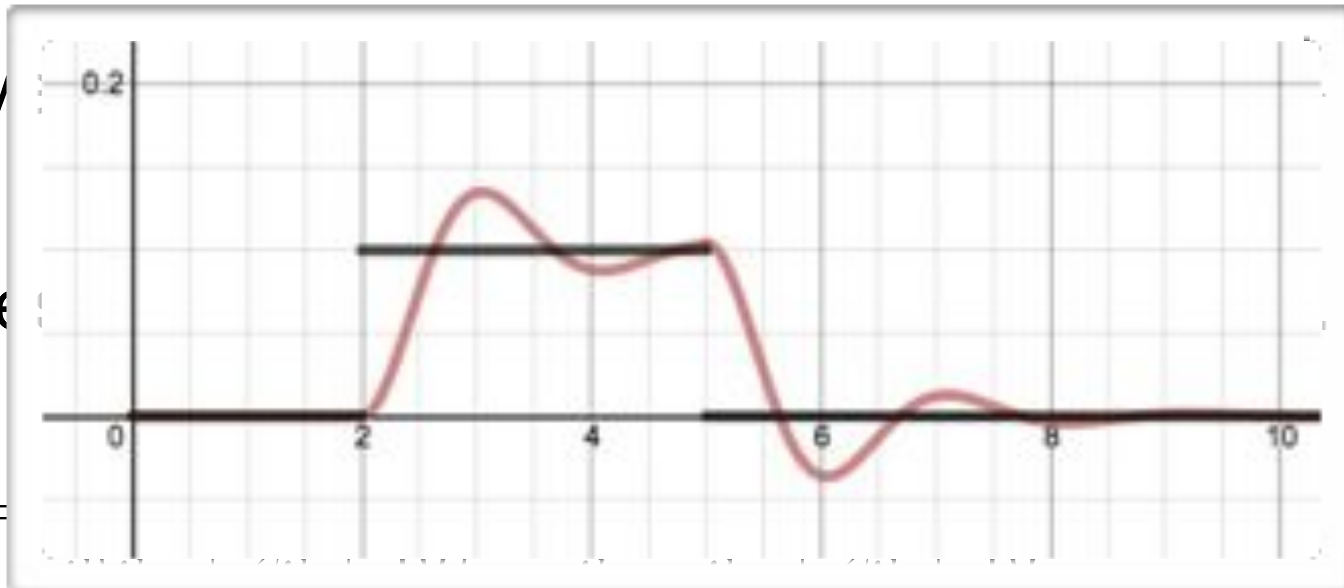

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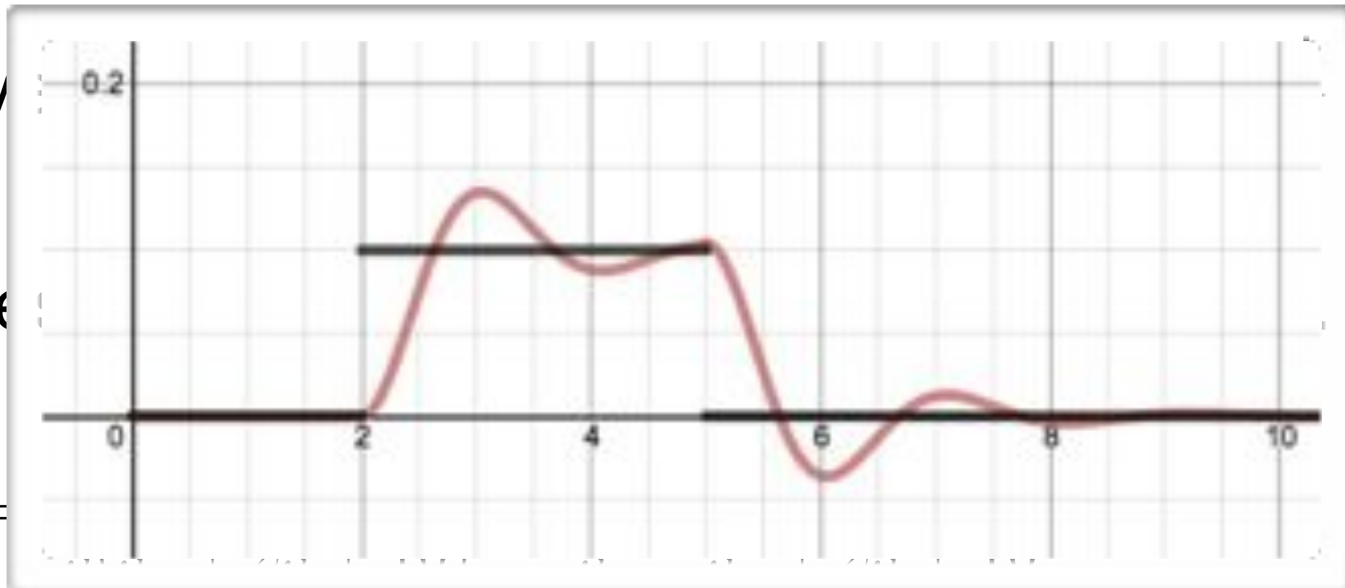

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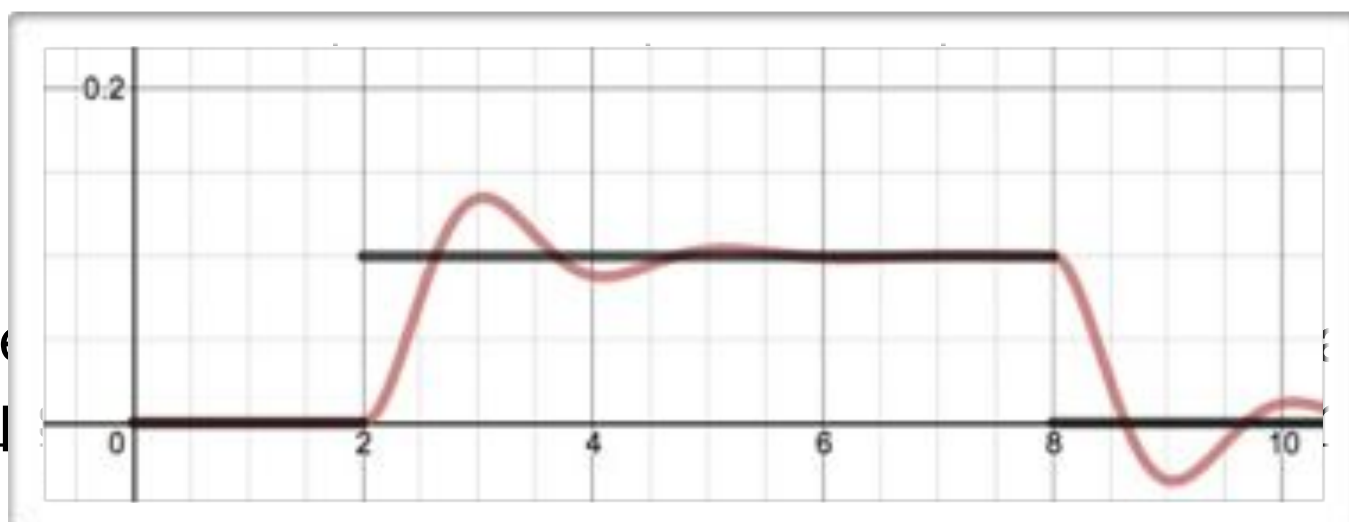

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