Webwork 5 - due Thursday

6 - will be posted soon

Tutorial Quiz - Monday on this week material.

Example

$$\vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -1/2 \end{pmatrix} \vec{x}$$

tr(A) = sum of diagonal entries

Look for eigenvalues:

$$r^2 - \operatorname{tr}(A) r + \det(A) = 0$$

$$r^2 + 1 r + 5/4 = 0$$

$$r = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-5}$$

$$r = -\frac{1}{2} \pm i$$

Look for eigenvectors

$r = -\frac{1}{2} + i$

eigenvector

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

I do not need the other eigenvector b/c I can use ~~complex conju~~ real and imaginary parts . . .

I have solution

$$\vec{x}_1 = e^{(-\frac{1}{2}+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= e^{-\frac{1}{2}t}\begin{pmatrix} \cos t + i \sin t \\ i\cos t - \sin t \end{pmatrix}$$

Set one solution = real part of this
other solution = im part of this.

Solution: $\vec{x} = c_1 e^{-\frac{1}{2}t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

Notice: write solution as real objects

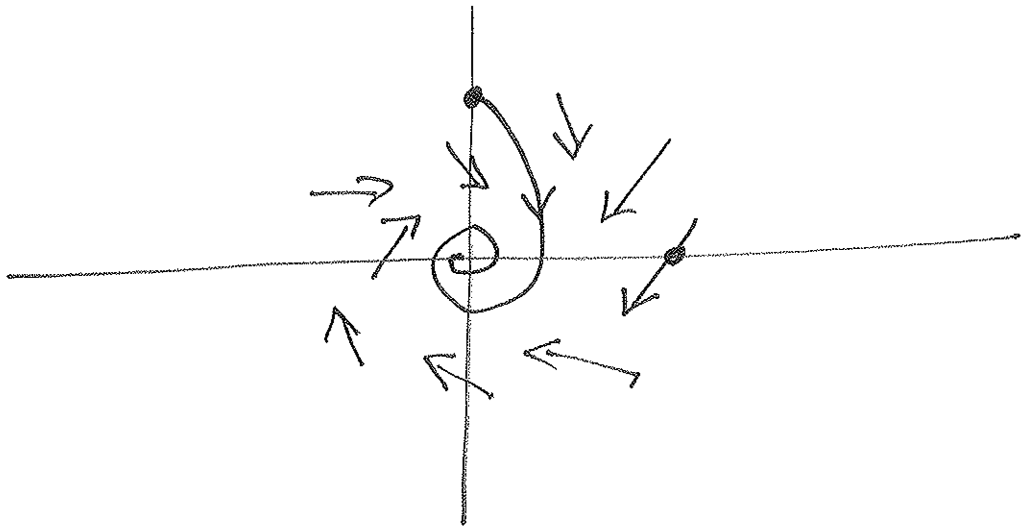
This is an <u>inward</u> spiral

b/c we have $e^{-\frac{1}{2}t}$

Generally, the factor $+a$ in e^{+at} is given by $\underline{\frac{tr(A)}{2}}$. <u>The sign of the trace</u> determines growth/decay of solutions (when you have complex eigenvalues

Vector field / solution curve for

$$\vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}$$

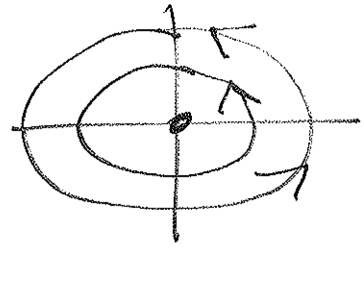


$$\begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \end{pmatrix}$$

If the eigenvalue $r = \lambda \pm i\mu$, then if $\lambda > 0$, you have an unstable spiral point at $(0,0)$.

if $\lambda < 0$, you have a stable spiral point.

If $\lambda = 0$, you have solutions like $e^{0t} e^{i\mu t}$, which neither grow nor decay - and the solution curves are elliptical.



We call $(0,0)$ a centre point in this case.

Repeated Eigenvalues

Consider 2×2 matrix with eigenvalues

$$r_1 = r_2 = r$$

Two cases:

(i) (easy) you have two distinct eigenvectors $\rightarrow$ proceed as usual.

e.g. $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$ $\quad r = 2$ eigenvalue

eigenvector: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

e.g. $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

could be the eigenvectors.

Solution: $\vec{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(ii) (tricky) you have only one eigenvector.
(Defective matrix case)

e.g. $\quad A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

Eigenvalues: $\quad r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$\underline{\underline{r=2}}$ single eigenvalue.

Eigenvector: $\quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Defective - one eigenvector.

So if we wanted to solve $\vec{x}' = A\vec{x}$ we are stuck.

We have $\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ but we need another.

Previously, multiplying by t helped.

So try $\vec{x}_2 = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Plug in to $\vec{x}' = A\vec{x}$.

$$\cancel{e^{2t}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2t\cancel{e^{2t}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = At\cancel{e^{2t}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \underline{\text{problem}}.$$

$$\times$$

Try instead

$$\vec{x}_2 = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\vec{d}$$

Plug in:

$$\cancel{e^{2t}}\begin{pmatrix}1\\-1\end{pmatrix} + \cancel{2te^{2t}\begin{pmatrix}1\\-1\end{pmatrix}} + 2\cancel{e^{2t}}\vec{d}$$

$$= \cancel{A\,te^{2t}\begin{pmatrix}1\\-1\end{pmatrix}} + A\cancel{e^{2t}}\vec{d}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} + 2\vec{d} = A\vec{d}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} = A\vec{d} - 2\vec{d}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} = (A - 2I)\vec{d}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1 & -1\\1 & 1\end{pmatrix}\underbrace{\begin{pmatrix}-1\\0\end{pmatrix}}_{\vec{d}\,.}$$

$\vec{x}_2 = te^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + e^{2t}\begin{pmatrix}-1\\0\end{pmatrix}$ is a solution

General sol$^{\underline{n}}$:

$$\vec{x} = c_1 e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2\left(te^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + e^{2t}\begin{pmatrix}-1\\0\end{pmatrix}\right)$$

Procedure: (Defective matrices)

if you have $r_1 = r_2 = r$
and one eigenvector $\vec{c}$.

1. obtain first solution $\vec{x} = e^{rt}\vec{c}$.
2. guess $\vec{x}_2 = te^{rt}\vec{c} + e^{rt}\vec{d}$

and plug in. Obtain an equation for $\vec{d}$ that should look like

$$(A - rI)\vec{d} = \vec{c}.$$

Solve for $\vec{d}$ (not unique solution)

3. general solution is

$$\vec{x} = B_1 e^{rt}\vec{c} + B_2\left(te^{rt}\vec{c} + e^{rt}\vec{d}\right)$$

where B_1, B_2 are constants.

$\vec{d}$ solves $(A - rI)\vec{d} = \vec{c}$ is called a generalized eigenvector of A.

Note: sign of r controls long-term growth or decay of the solution.

Recall: $te^{-wt} \to 0$ as $t \to \infty$ for any $w > 0$.

Example $\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$

Eigenvalues $r^2 - 2r + 1 = 0$

$$(r-1)^2 = 0$$

$$r = 1$$

Eigenvector $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution : $\vec{x}_1 = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Build $\vec{x}_2 = te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \vec{d}$

Plug in : ...

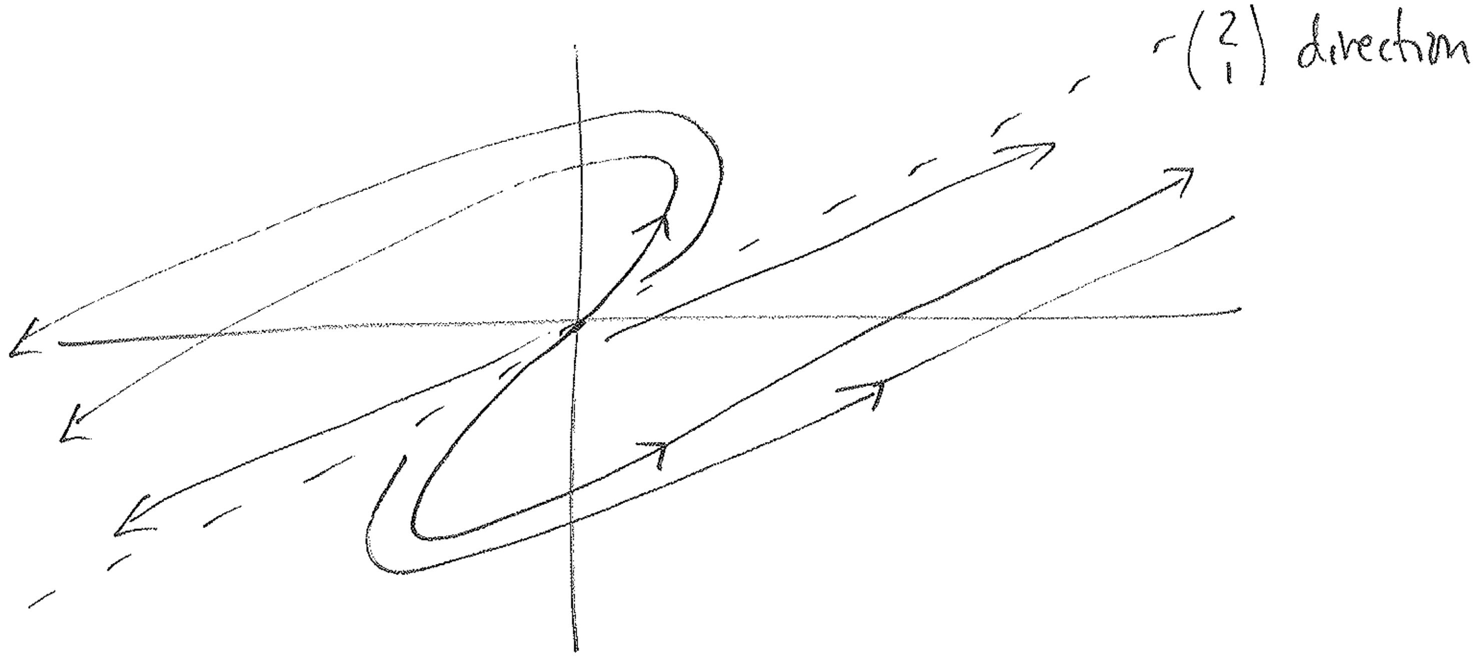
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \overset{\vec{d_2}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\left[\ \vec{c} = (A - rI)\vec{d} \ \right]$$

General Solution:

$$\vec{x} = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

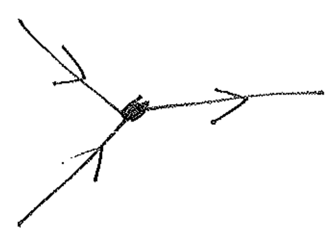
Notice: as $t \to \infty$, solution approaches the $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ direction.



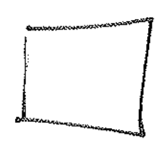
Example Application of systems of 1st order ODE to electric circuits

Kirchhoff laws

(i) net current at a point $= 0$



(ii) net voltage drop around a loop $= 0$



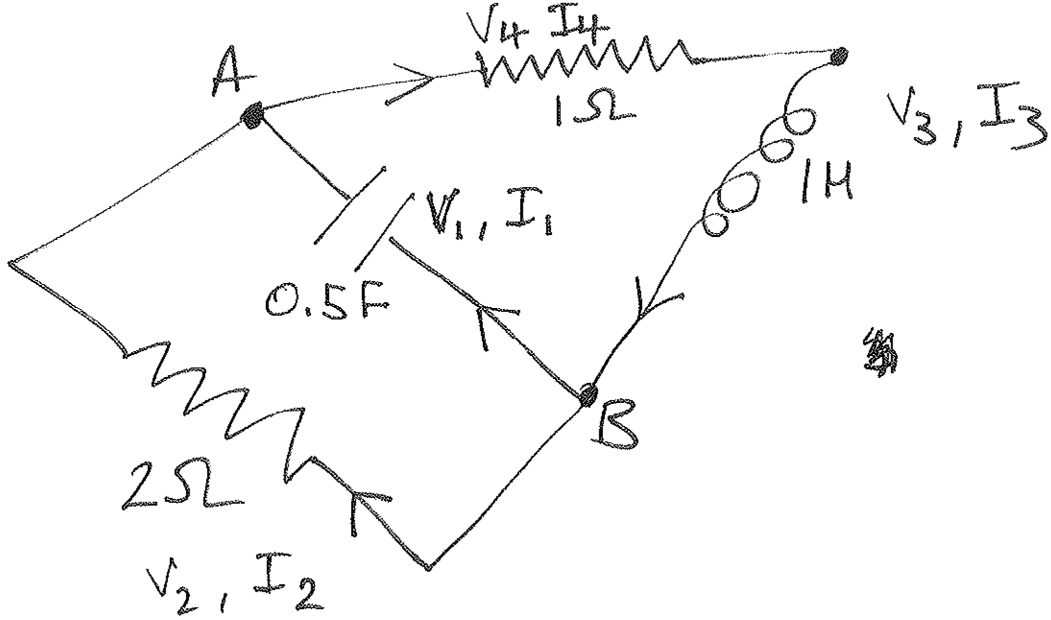
Component rules:

Resistor : $V = IR$

Capacitor : $V = \dfrac{Q}{C} \rightarrow \dfrac{dV}{dt} = \dfrac{1}{C}\dfrac{dQ}{dt} = \dfrac{1}{C} I$

Inductor : $L\dfrac{dI}{dt} = V$

Any circuit built with these components is equivalent to a linear system of ODE.



Components:

$$V_2 = 2I_2$$
$$V_4 = I_4 = I_3$$
} resistors

$$\frac{1}{2}\frac{dV_1}{dt} = I_1$$
} capacitor

$$\frac{dI_3}{dt} = V_3$$
} inductor

at A:

$$I_1 + I_2 - I_3 = 0$$

at B:

$$I_3 - I_2 - I_1 = 0$$

lower loop:

$$V_1 = V_2$$

upper loop:

$$V_1 + V_4 + V_3 = 0$$

(eliminate, simplify)

$$\frac{d}{dt}\begin{pmatrix} I_3 \\ V_1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} I_3 \\ V_1 \end{pmatrix}$$

Analyze: $r^2 + 2r + 3 = 0$

$$r = -1 \pm \sqrt{2}\, i$$

Eigenvectors: $\begin{pmatrix} -\sqrt{2}\,i & -1 \\ 2 & -\sqrt{2}\,i \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Take real + im parts of $e^{-t} e^{\sqrt{2}it} \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix}$

$$= e^{-t} \begin{pmatrix} \cos\sqrt{2}t + i \sin\sqrt{2}t \\ -\sqrt{2}\,i \cos\sqrt{2}t + \sqrt{2} \sin\sqrt{2}t \end{pmatrix}$$

Solution: $\begin{pmatrix} I_3 \\ V_1 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{pmatrix}$

Suppose $I_3(0) = 0$
$V_1(0) = 2$ volts

Solve for c_1 and c_2: get $\begin{pmatrix} I_3 \\ V_1 \end{pmatrix} = \sqrt{2}e^{-t} \begin{pmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{pmatrix}$

Plot current I_3