Today

- Step and ramp functions (continued)
- The Dirac Delta function and impulse force
- (Modeling with delta-function forcing)

Step function forcing

- Solve using Laplace transforms:

$$y'' + 2y' + 10y = g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

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- So we just need h(t) and we're done.

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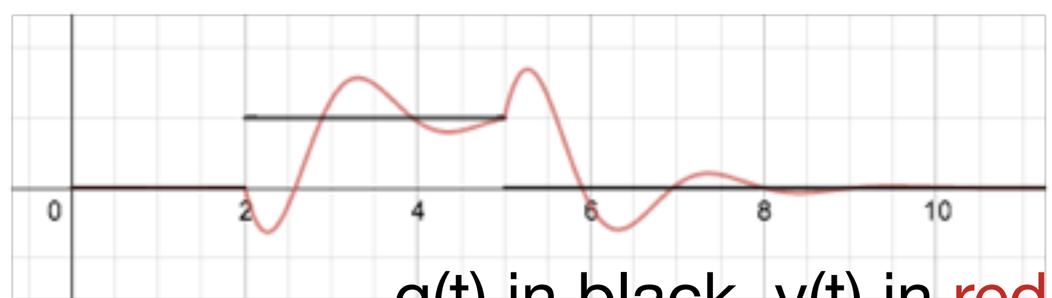

g(t) in black, y(t) in red.

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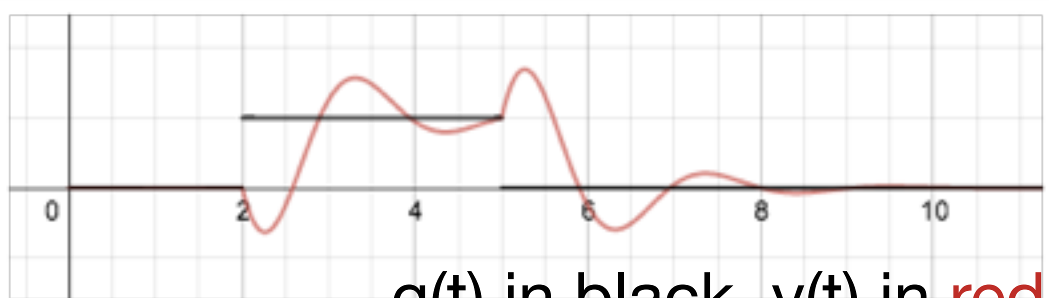

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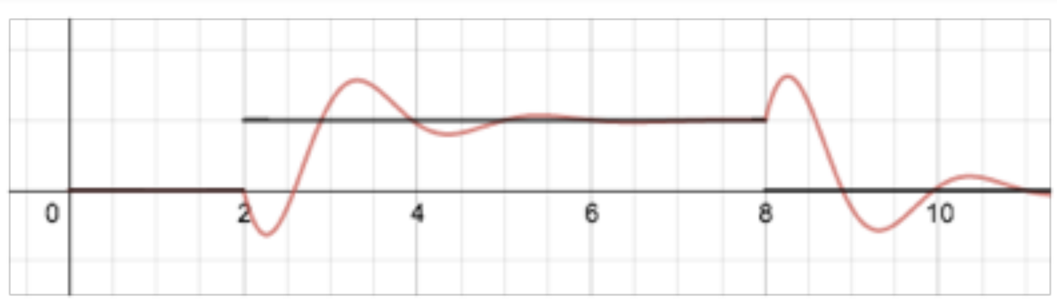

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Step function forcing

- An example with a ramped forcing function:

$$y'' + 4y = \begin{cases} 0 & \text{for } t < 5, \\ \frac{t-5}{5} & \text{for } 5 \leq t < 10, \\ 1 & \text{for } t \geq 10. \end{cases}$$

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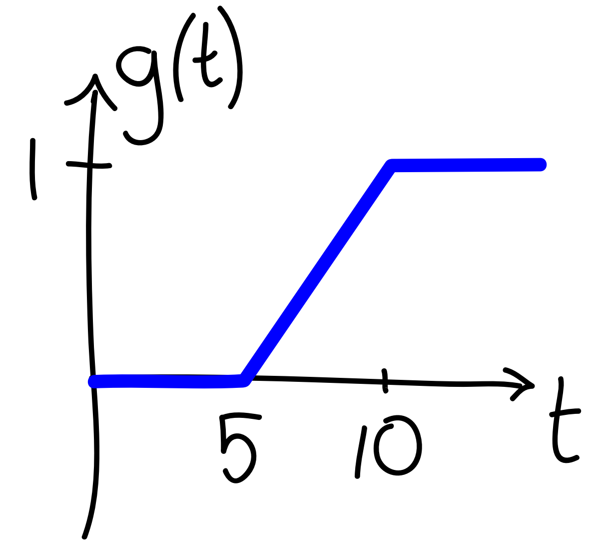

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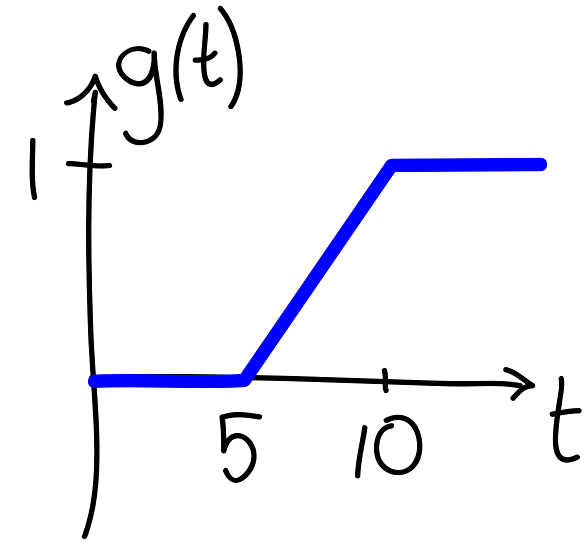

- Write g(t) is terms of $u_c(t)$:

(A) $g(t) = u_5(t) - u_{10}(t)$

(B) $g(t) = u_5(t)(t-5) - u_{10}(t)(t-5)$

(C) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/5$

(D) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/10$

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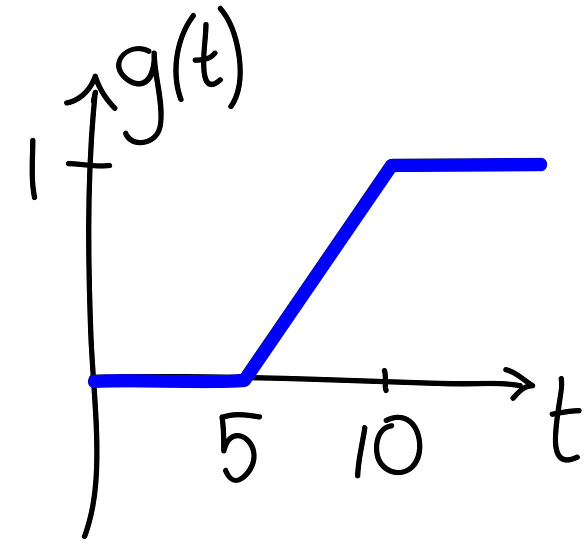

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Two methods:

1. Build from left to right, adding/subtracting what you need to make the next section:

$$g(t) = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

2. Build each section independently:

$$g(t) = \left(u_5(t) - u_{10}(t)\right)\frac{1}{5}(t-5) + u_{10}(t)\cdot 1$$

- W

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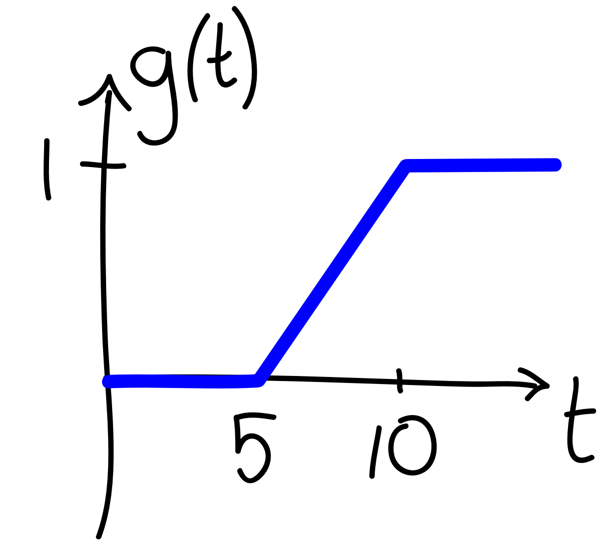

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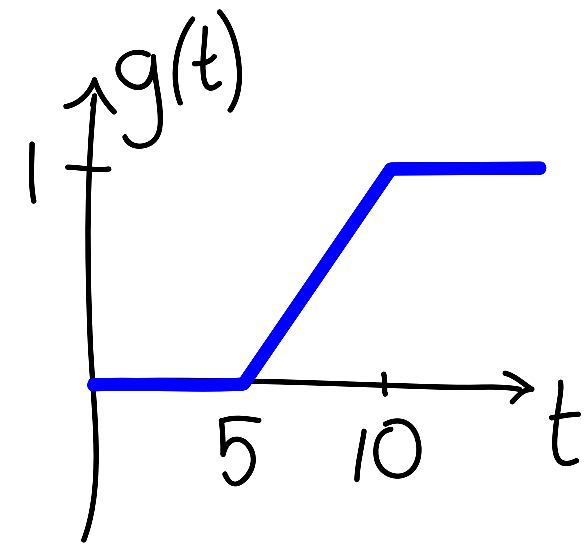

$$s^2Y + 4Y =$$

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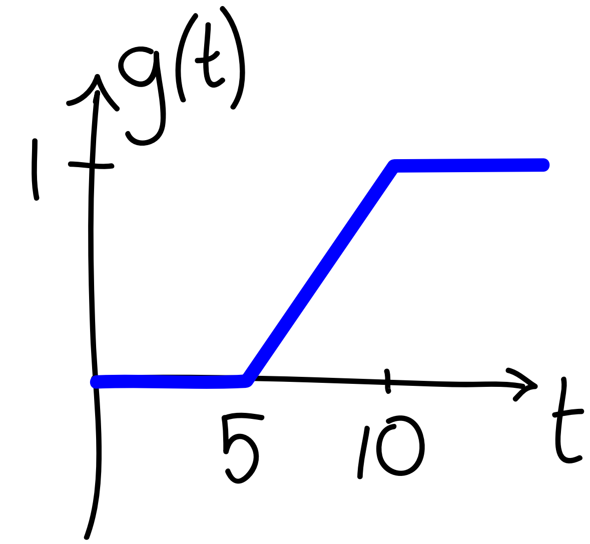

$$s^2Y + 4Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

$$Y(s) = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)} \quad = \frac{1}{5}(e^{-5s} - e^{-10s})H(s)$$

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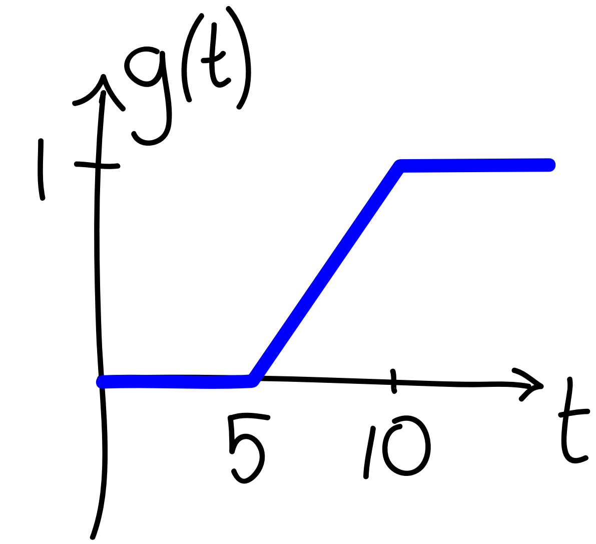

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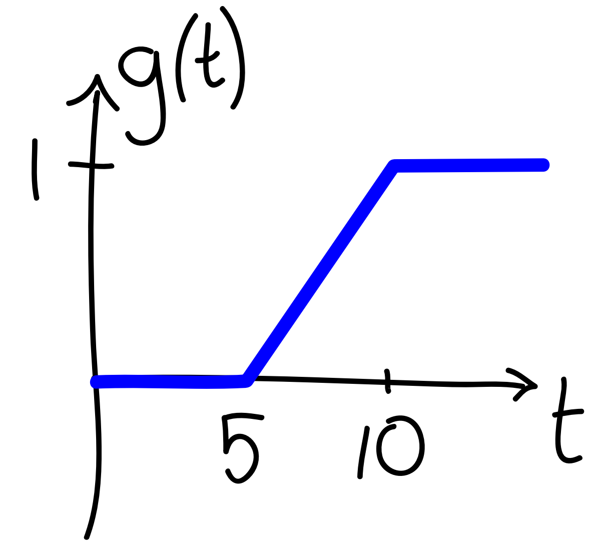

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$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

Delta-function forcing

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- Suppose a mass is sitting at position x and a force g(t) acts on it:

$$mx'' = g(t)$$

- To find x(t), integrate up:

$$\int_a^b mx'' \; dt = \int_a^b g(t) \; dt$$

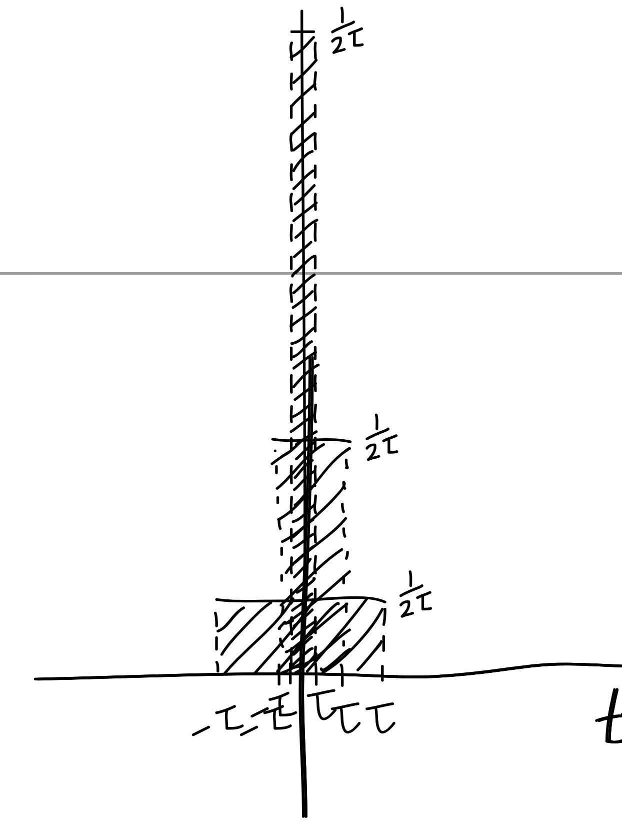
$$mx' \Big|_a^b = \int_a^b g(t) \; dt$$

$$mv(b) - mv(a) = \int_a^b g(t) \; dt$$

- $\int_a^b g(t) \; dt$ is the change in momentum of the mass - called impulse.

- If the force is large and sudden (say a hammer hitting the mass), maybe we just need to get this integral correct and the details don't matter.

Delta-function forcing



- Let's assume $g(t) = \begin{cases} \dfrac{I_0}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$

$$= (u_{-\tau}(t) - u_\tau(t))\frac{I_0}{2\tau}$$

impulse = $\Delta\text{momentum} = \int_{-\infty}^{\infty} g(t)\; dt \;= \int_{-\tau}^{\tau} \frac{I_0}{2\tau}\; dt \;= I_0$

- For general purposes (any property that might change quickly, not just momentum), we define the Dirac Delta "function" as follows:

$$d_\tau(t) = (u_{-\tau}(t) - u_\tau(t))\frac{1}{2\tau}$$

$$\delta(t) = \lim_{\tau\to 0} d_\tau(t) = \begin{cases} \text{“}\infty\text{”} & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases}$$

$$g(t) = I_0 d_\tau(t)$$

- I_0 can be replaced by any type of quantity
- e.g. m_0 mass added to tank suddenly
- units of δ(t): 1 / time

Some facts about the Delta "function"

$$\int_a^b \delta(t)\ dt = 1 \qquad a < 0,\ b > 0$$ and = 0 otherwise.

$$\int_a^b f(t)\delta(t)\ dt = \lim_{\tau \to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(t)\ dt$$

$$= \lim_{\tau \to 0} \frac{F(\tau) - F(-\tau)}{2\tau} \qquad F'(t) = f(t)$$

$$= F'(0)\ = f(0)$$

$$\int_a^b f(t)\delta(t)\ dt = f(0) \qquad a < 0,\ b > 0$$ and = 0 otherwise.

$\delta(t - c) =$ shift of $\delta(t)$ by c

$$\int_a^b f(t)\delta(t-c)\ dt\ = \int_{a+c}^{b+c} f(u+c)\delta(u)\ du\ = f(c)$$ provided a<c<b.

Some facts about the Delta "function"

$$\int_a^b f(t)\delta(t-c)\ dt\ = f(c)$$

Laplace transform of delta function:

$$\mathcal{L}\{\delta(t-c)\} = \int_0^\infty e^{-st}\delta(t-c)\ dt$$

$$= \int_{-c}^\infty e^{-s(u+c)}\delta(u)\ du\ \ = e^{-sc} \text{ for } c > 0$$

Relationship of delta function to other functions:

$$\frac{d}{dt}|t-c| = \ 2u_c(t) - 1$$

$$\frac{d}{dt}u_c(t) = \ \ \delta(t-c)$$