Today

- Finish up undetermined coefficients
- Physics applications - mass springs
- Undamped, over/under/critically damped oscillations

Method of undetermined coefficients (3.5)

- Example. Find the general solution to $y'' + 2y' = e^{2t} + t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt$

(B) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt + E$

(C) $y_p(t) = Ae^{2t} + (Bt^4 + Ct^3 + Dt^2 + Et)$
$y_p(t) = Ae^{2t} + t(Bt^3 + Ct^2 + Dt + E)$

(D) $y_p(t) = Ae^{2t} + Be^{-2t} + Ct^3 + Dt^2 + Et + F$

(E) Don't know / still thinking.

For each wrong answer, for what DE is it the correct form?

Method of undetermined coefficients (3.5)

- Example. Find the general solution to $y'' - 4y = t^3 e^{2t}$.
  - What is the form of the particular solution?

(A) $y_p(t) = (At^3 + Bt^2 + Ct + D)e^{2t}$

(B) $y_p(t) = (At^3 + Bt^2 + Ct)e^{2t}$

(C) $y_p(t) = (At^3 + Bt^2 + Ct)e^{2t} + (Dt^3 + Et^2 + Ft)e^{-2t}$

★ (D) $y_p(t) = (At^4 + Bt^3 + Ct^2 + Dt)e^{2t}$
$y_p(t) = t(At^3 + Bt^2 + Ct + D)e^{2t}$

(E) Don't know / still thinking.

Method of undetermined coefficients (3.5)

$$y'' + 3y' - 10y = x^2 e^{-5x}$$

$$y_h(x) = C_1 e^{-5x} + C_2 e^{2x}$$

$$y = e^{rx}$$
$$r^2 + 3r - 10 = 0$$
$$r = -5, 2$$

$$y_p(x) = Ax^2 e^{-5x}$$
$$y_p'(x) = 2Axe^{-5x} - 5Ax^2 e^{-5x}$$
$$y_p''(x) = 2Ae^{-5x} - 10Axe^{-5x} - 10Axe^{-5x} + 25Ax^2 e^{-5x}$$

$$\begin{aligned}
-10y_p(x) &= && - 10Ax^2 e^{-5x} \\
3y_p'(x) &= & 6Axe^{-5x} & - 15Ax^2 e^{-5x} \\
y_p''(x) &= 2Ae^{-5x} & - 20Axe^{-5x} & + 25Ax^2 e^{-5x} \\
\hline
& 2Ae^{-5x} & - 14Axe^{-5x} & + 0 \quad = x^2 e^{-5x}
\end{aligned}$$

Can’t find A that works! Need 3 unknowns to match all 3 terms.

Method of undetermined coefficients (3.5)

$$y'' + 3y' - 10y = x^2 e^{-5x}$$

$$y_h(x) = C_1 e^{-5x} + C_2 e^{2x}$$

$$y_p(x) = Ax^2 e^{-5x} + Bxe^{-5x} + Ce^{-5x}$$

$y_p'(x)$ involves $x^2, x, 1$

$y_p''(x)$ involves $x^2, x, 1$

But e^{-5x} gets killed by the operator so C disappears - only 2 unknowns for matching.

Need 3 unknowns but not including e^{-5x}.

$$\begin{aligned} y_p(x) &= Ax^3 e^{-5x} + Bx^2 e^{-5x} + Cxe^{-5x} \\ &= x(Ax^2 e^{-5x} + Bxe^{-5x} + Ce^{-5x}) \end{aligned}$$

Method of undetermined coefficients (3.5)

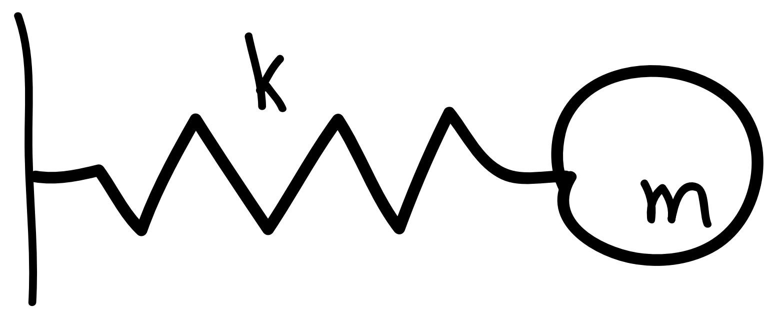
- Summary - finding a particular solution to L[y] = g(t).
  - Include all functions that are part of the g(t) family (e.g. cos **and** sin)
  - If part of the g(t) family is a solution to the homogeneous (h-)problem, use t x (g(t) family).
  - If t x (part of the g(t) family), is a solution to the h-problem, use t^2 x (g(t) family). etc.
  - For sums, group terms into families and include a term for each. You can even find a y_p for each family separately and add them up.
  - Works for products of functions - be sure to include the whole family!
  - Never include a solution to the h-problem as it won't survive L[ ]. Just make sure you aren't missing another term somewhere.

Method of undetermined coefficients (3.5)

- Do lots of these problems and the trends will become clear.
- Try different y_ps and see what goes wrong - this will help you see what must happen when things go right.
- Two crucial facts to remember
    - If you try a form and you can make LHS=RHS with some choice for the coefficients then you're done.
    - If you can't, your guess is most likely missing a term(s).

Applications - vibrations (3.7)

Mass-spring systems



$$E = \frac{1}{2}k(x-x_0)^2$$

$$F = -\frac{dE}{dx} = -k(x-x_0)$$

$$ma = F$$

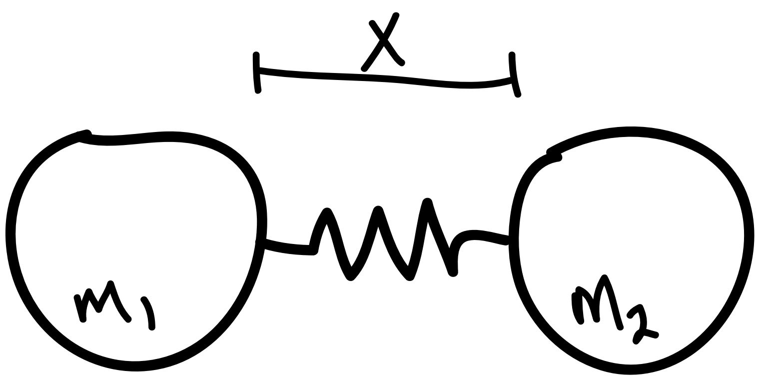
$$ma = -k(x-x_0)$$

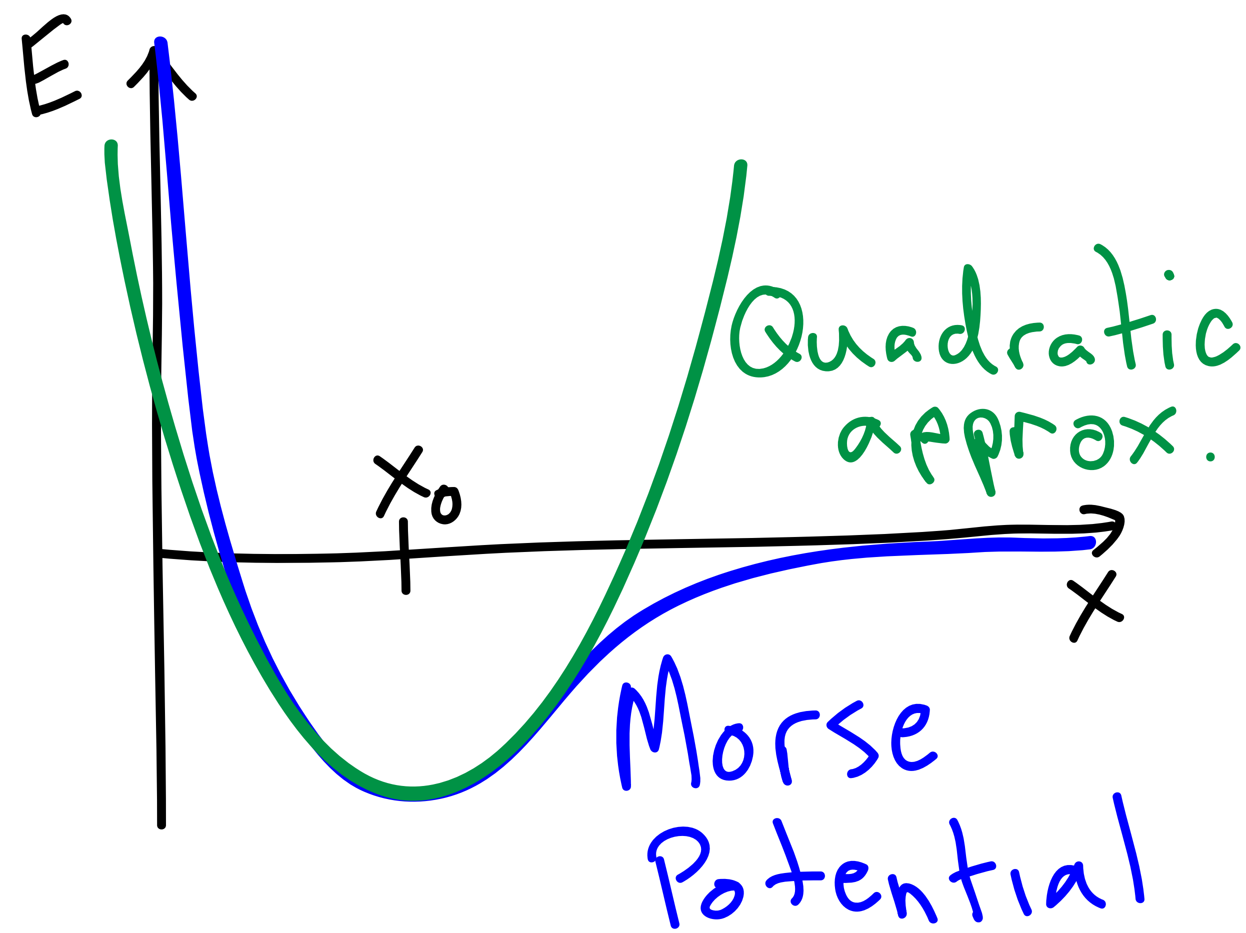
$$mx'' = -k(x-x_0)$$

$$mx'' + kx = kx_0$$

Applications - vibrations (3.7)

Molecular bonds

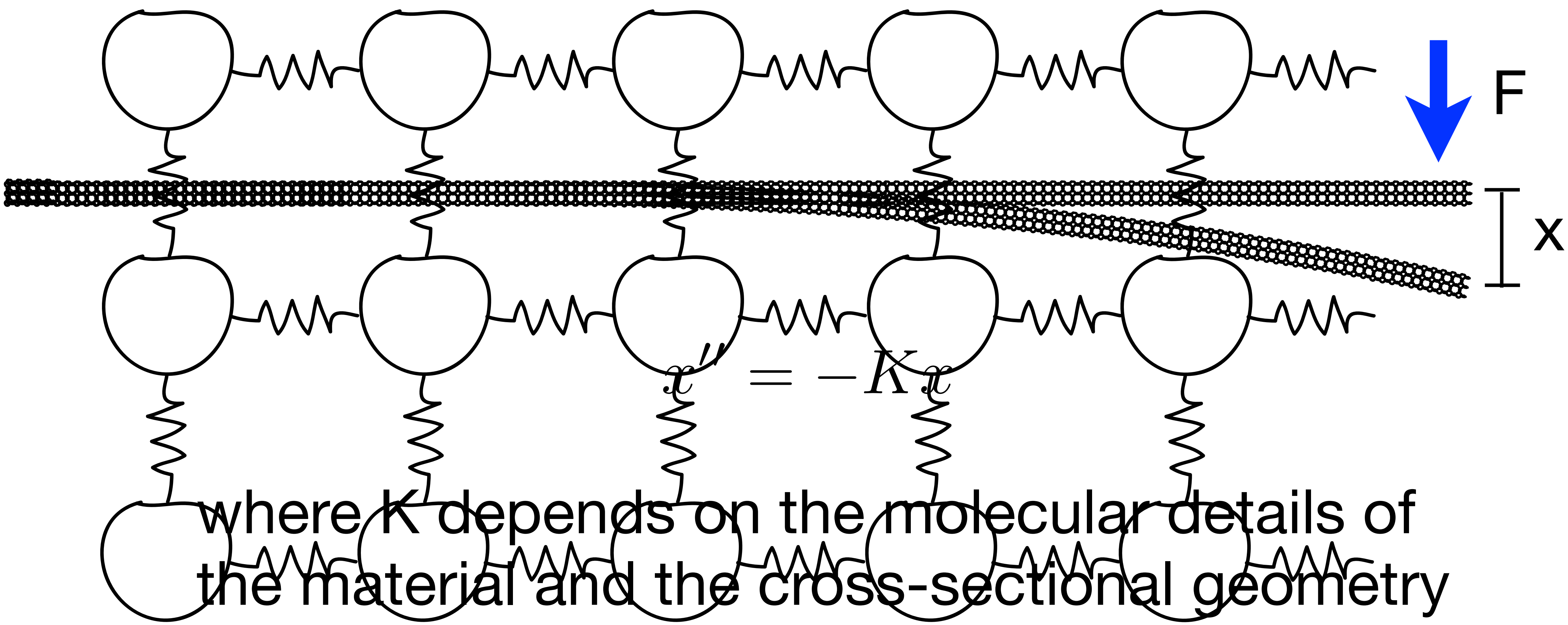




Applications - vibrations (3.7)

Solid mechanics

e.g. tuning fork, bridges, buildings



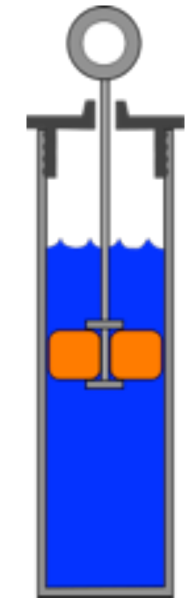
$$x'' = -Kx$$

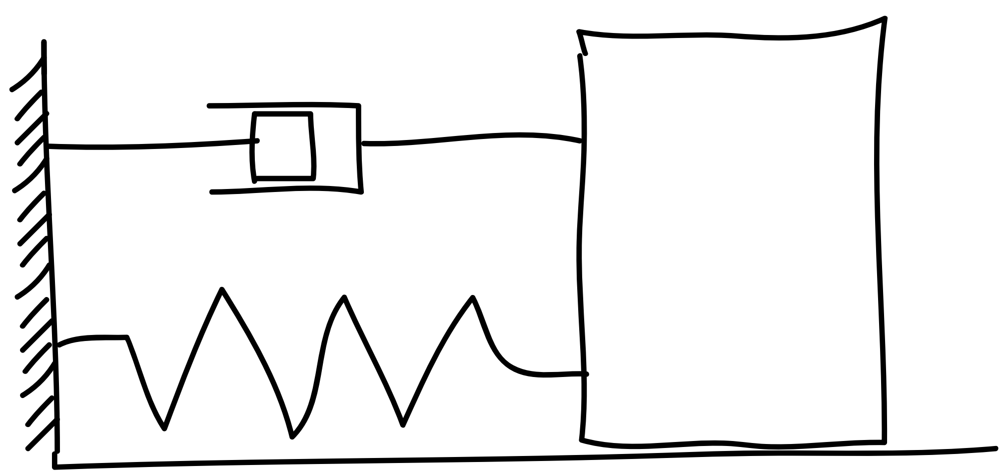
where K depends on the molecular details of the material and the cross-sectional geometry of the rod.

Applications - vibrations (3.7)

- So far, no x' term so no exponential decay in the solutions.

- Dashpot - mechanical element that adds friction.
  - sometimes an abstraction that accounts for energy loss.





Kelvin-Voigt model

shock absorber

$$ma = -k(x - x_0) - \gamma v$$

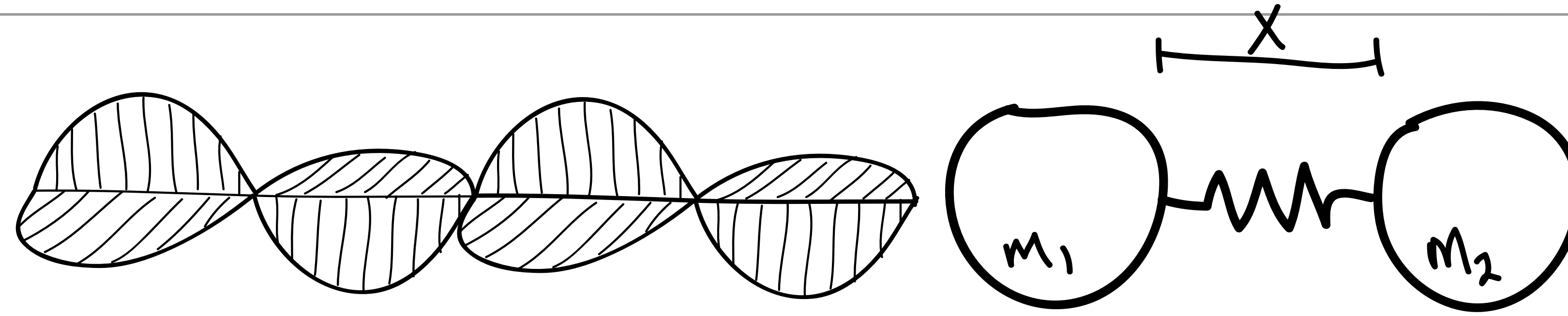
$$mx'' = -k(x - x_0) - \gamma x'$$

$$mx'' + \gamma x' + kx = kx_0$$

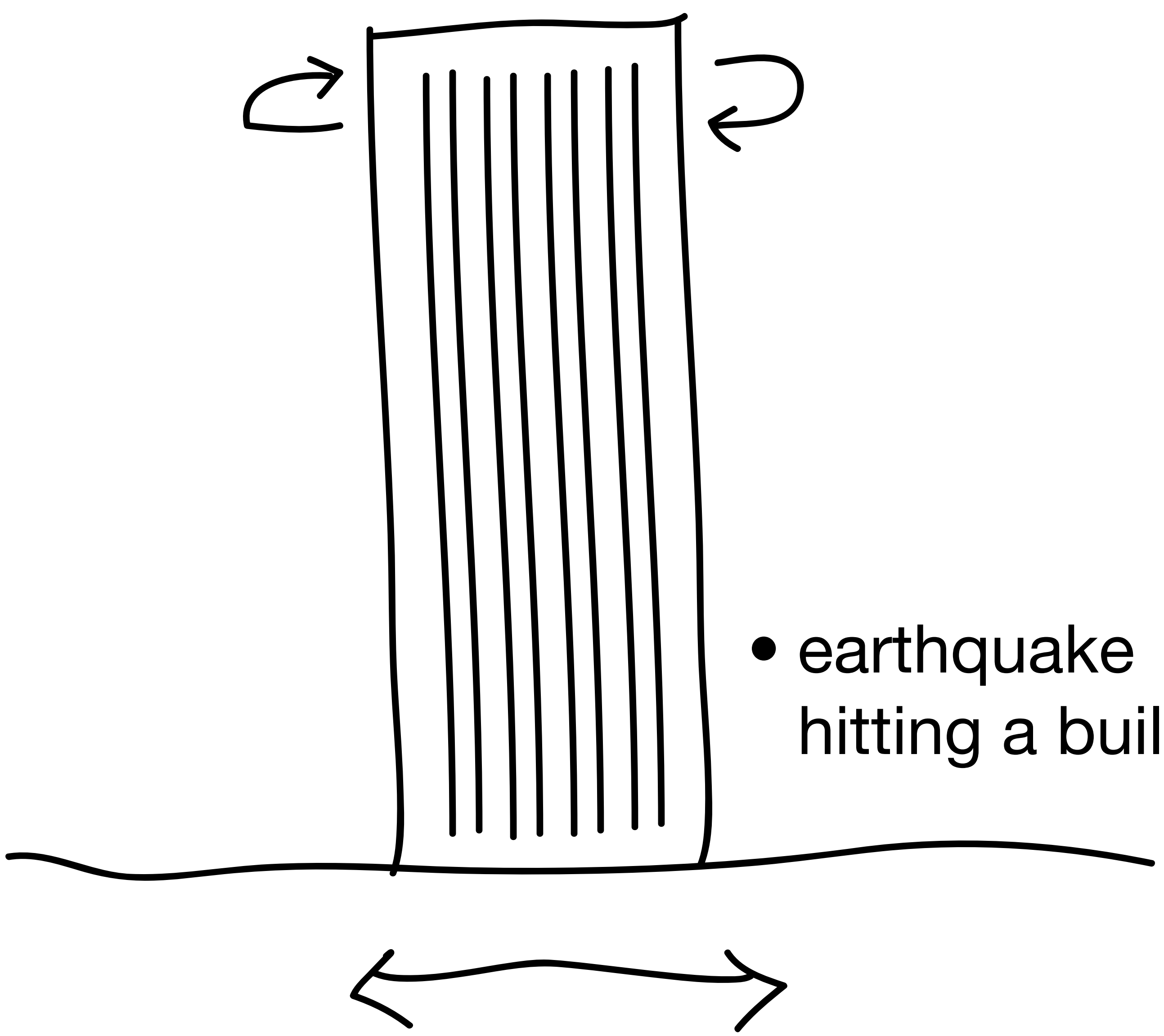
$$y = x - x_0$$

$$my'' + \gamma y' + ky = 0$$

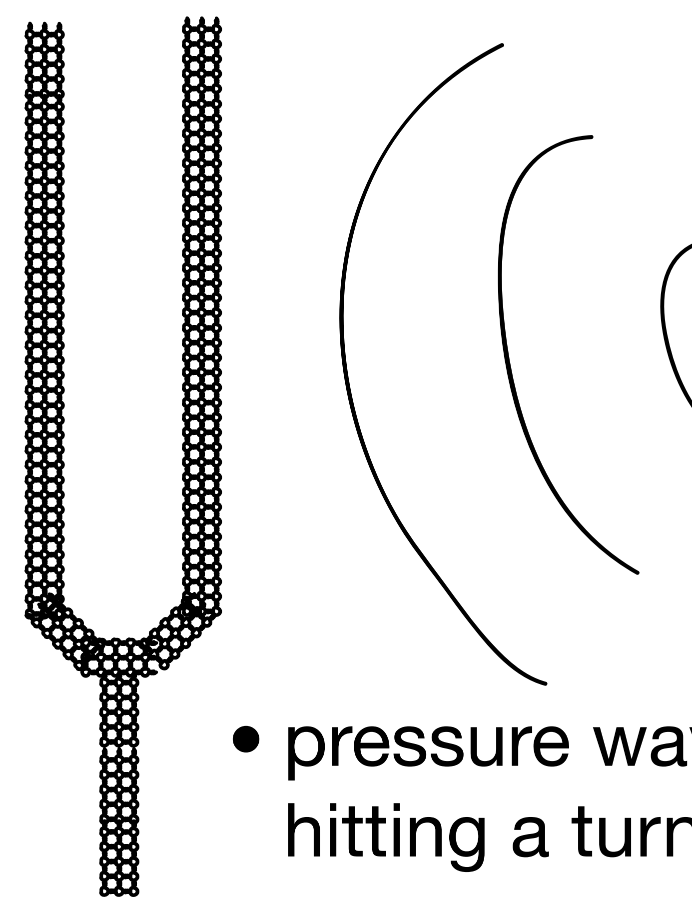
Applications - forced vibrations (3.8)



- light hitting a molecular bond



- earthquake hitting a building



- pressure waves (sound) hitting a turning fork.

Applications - vibrations (3.7)

- Undamped mass spring

$$mx'' + kx = 0$$

$$mr^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}i$$

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- frequency
  - increases with stiffness
  - decreases with mass

Applications - vibrations (3.7)

Trig identity reminders

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$2\cos(3t+\pi/3) =$$

(A) $2\sin(\pi/3)\cos(3t) - 2\sin(\pi/3)\cos(3t)$

(B) $2\sin(\pi/3)\cos(3t) + 2\sin(\pi/3)\cos(3t)$

(C) $2\cos(\pi/3)\cos(3t) - 2\sin(\pi/3)\sin(3t)$

(D) $2\cos(\pi/3)\cos(3t) + 2\sin(\pi/3)\sin(3t)$

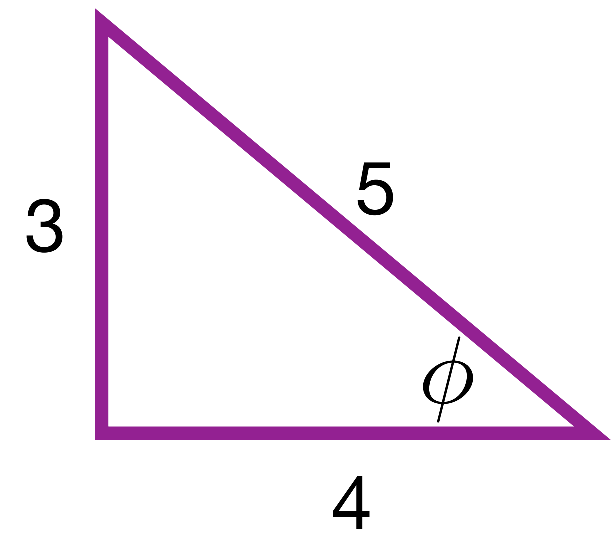
$= \cos(3t) - \sqrt{3}\sin(3t)$

(E) Don't know / still thinking.

Applications - vibrations (3.7)

- Converting from sum-of-sin-cos to a single cos expression:
  - Example:

$$
\begin{aligned}
&4\cos(2t) + 3\sin(2t) \\
&\quad = 5\left(\frac{4}{5}\cos(2t) + \frac{3}{5}\sin(2t)\right) \\
&\quad = 5(\cos(\phi)\cos(2t) + \sin(\phi)\sin(2t)) \\
&\quad = 5\cos(2t - \phi)
\end{aligned}
$$



$$\phi = 0.9273$$

$$\cos(A - B) = \overset{\cancel{4}}{\cos(A)}\cos(B) + \overset{\cancel{3}}{\sin(A)}\sin(B)$$

(cos(A), sin(A)) must lie on the unit circle. i.e. $\cos^2(A)+\sin^2(A) = 1$.

Applications - vibrations (3.7)

- Converting from sum-of-sin-cos to a single cos expression:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

  - Step 1 - Factor out $A = \sqrt{C_1^2 + C_2^2}$.
  - Step 2 - Find the angle ϕ for which $\cos(\phi) = \dfrac{C_1}{\sqrt{C_1^2 + C_2^2}}$ and $\sin(\phi) = \dfrac{C_2}{\sqrt{C_1^2 + C_2^2}}$.
  - Step 3 - Rewrite the solution as $y(t) = A\cos(\omega_0 t - \phi)$.

Applications - vibrations (3.7)

- Damped mass-spring

$$mx'' + \gamma x' + kx = 0 \qquad\qquad m, \gamma, k > 0$$

$$\Rightarrow \quad mr^2 + \gamma r + k = 0$$

$$r_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4km}}{2m} \quad = \underbrace{\frac{\gamma}{2m}\left(-1 \pm \underbrace{\sqrt{1 - \frac{4km}{\gamma^2}}}_{\text{smaller than 1 or complex}}\right)}_{\text{negative or complex}}$$

We have the usual three cases...

Applications - vibrations (3.7)

- Damped oscillations

$$r_{1,2} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

(i) $\frac{4km}{\gamma^2} < 1 \quad \Rightarrow$ $r_1, r_2 < 0$, exponential decay only

(over damped - γ large)

(ii) $\frac{4km}{\gamma^2} = 1 \quad \Rightarrow$ $r_1 = r_2$, exp and t*exp decay

(critically damped)

(iii) $\frac{4km}{\gamma^2} > 1 \quad \Rightarrow \quad r = \alpha \pm \beta i$

$$\alpha = -\frac{\gamma}{2m} < 0 \Rightarrow$$ decaying oscillations

(under damped - γ small)

$$x(t) = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

$$\beta = \sqrt{\frac{4km}{\gamma^2} - 1}$$

called pseudo-frequency

For graphs, see:

https://www.desmos.com/calculator/psy5r8hpln