Today

- Non-homogeneous systems of ODEs
- Non-homogeneous two-tank example
- Intro to Laplace transforms

Nonhomogeneous system of DEs

- How do you solve the equation

$$\mathbf{x}'(\mathbf{t}) = A\mathbf{x}(\mathbf{t}) + \mathbf{b} \text{ ?}$$

- Define the linear operator

$$L[\mathbf{x}] = \mathbf{x}'(\mathbf{t}) - A\mathbf{x}(\mathbf{t})$$

- The equation above can be written as

$$L[\mathbf{x}] = \mathbf{b}$$

- As for 2nd order equations, solve homogeneous eqn first,

$$L[\mathbf{x}] = \mathbf{0}$$

- then Method of Undetermined Coefficients...

Nonhomogeneous system of DEs

- For the equation,

$$\mathbf{x}'(\mathbf{t}) = A\mathbf{x}(\mathbf{t}) + \mathbf{b}$$

which of the following is a suitable guess (in the sense of MUC)?

(A) $\mathbf{x_p} = c\mathbf{b}$ -- works only when **b** happens to be an eigenvector associated with a non-zero eigenvalue; not really worth trying.

(B) $\mathbf{x_p} = \mathbf{v}$ -- works when **b** is in the range of A (which is to say often so try this first).

(C) $\mathbf{x_p} = t\mathbf{v}$ -- works when (B) doesn't and **b** happens to be in the nullspace of A.

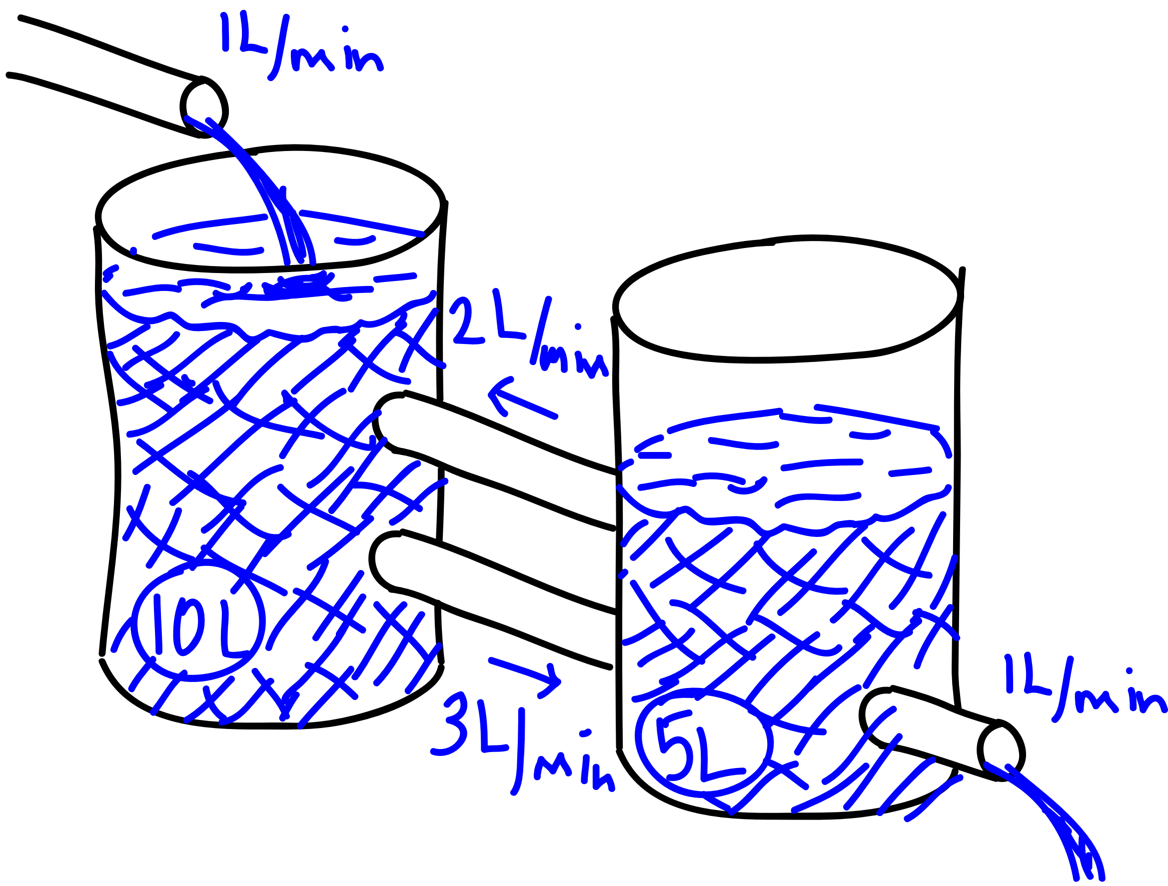
(D) $\mathbf{x_p} = t\mathbf{u} + \mathbf{v}$ -- works when (B) and (C) don't with one exception but is beyond the scope of this course.

(E) Huh?

Nonhomogeneous system of DEs - example

- Salt water flows into a tank holding 10 L of water at a rate of 1 L/min with a concentration of 200 g/L. The well-mixed solution flows from that tank into a tank holding 5 L through a pipe at 3 L/min. Another pipe takes the solution in the second tank back into the first at a rate of 2 L/min. Finally, solution drains out of the second tank at a rate of 1 L/min.
- Write down a system of equations in matrix form for the mass of salt in each tank.

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$



Nonhomogeneous case - example

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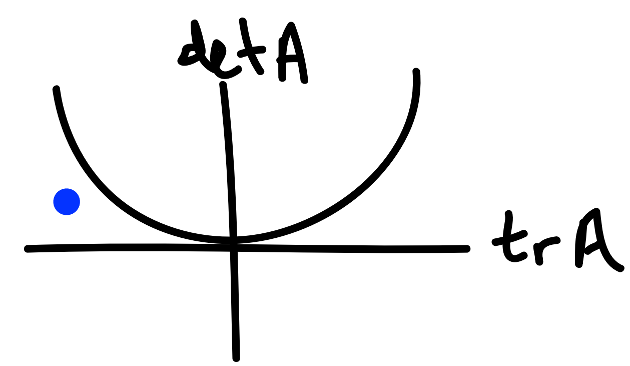
- Find the eigenvalues and the long term (steady state) solution.

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

$$\text{tr}A = -\frac{9}{10} \qquad (\text{tr}A)^2 = \frac{81}{100}$$

$$\det A = \frac{9}{50} - \frac{6}{50} = \frac{3}{50} \qquad 4\det A = \frac{12}{50}$$



Both evalues negative!

Nonhomogeneous case - example

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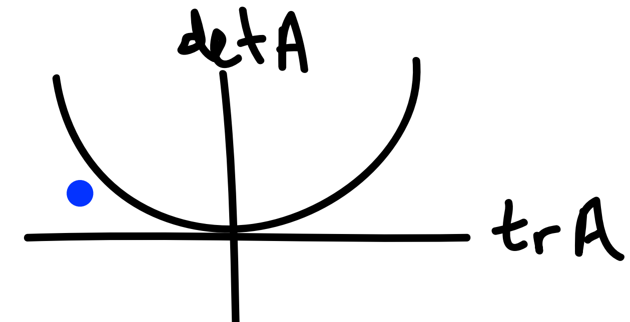


Both evalues negative!

$$\mathbf{m_h}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} \qquad \left(\lambda_{1,2} = -\frac{9}{20} \pm \frac{\sqrt{57}}{20} \right)$$

$$\mathbf{m_p}(t) = \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\mathbf{0} = A\mathbf{w} + \begin{pmatrix} 200 \\ 0 \end{pmatrix} \rightarrow A\mathbf{w} = -\begin{pmatrix} 200 \\ 0 \end{pmatrix} \rightarrow \mathbf{w} = \begin{pmatrix} 2000 \\ 1000 \end{pmatrix}$$

$$\mathbf{m}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} + \begin{pmatrix} 2000 \\ 1000 \end{pmatrix}$$

Nonhomogeneous case - example

- A “Method of undetermined coefficients” similar to what we saw for second order equations can be used for systems.

- For this course, I’ll only test you on constant nonhomogeneous terms (like the previous example).

Laplace transforms - intro (6.1)

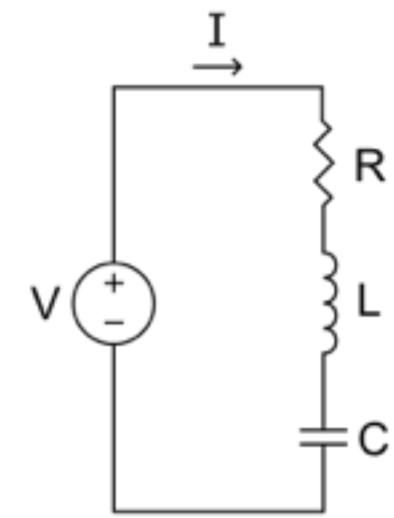
- Motivation for Laplace transforms:
  - We know how to solve $ay'' + by' + cy = g(t)$ when $g(t)$ is polynomial, exponential, trig.
  - In applications, $g(t)$ is often “piece-wise continuous” meaning that it consists of a finite number of pieces with jump discontinuities in between. For example,

$$g(t) = \begin{cases} \sin(\omega t) & 0 < t < 10, \\ 0 & t \geq 10. \end{cases}$$

  - These can be handled by previous techniques (modified) but it isn’t pretty (solve from t=0 to t=10, use y(10) as the IC for a new problem starting at t=10).

Laplace transforms - intro (6.1)

- Motivation for Laplace transforms - example RLC circuit

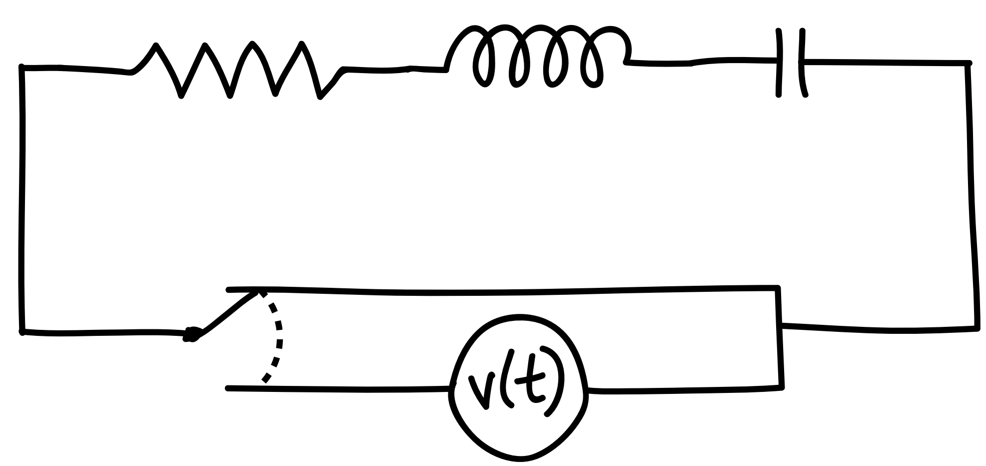


  - Resistor, inductor and capacitor in series

$$I''(t) + \frac{R}{L}I'(t) + \frac{1}{LC}I(t) = v(t)$$

  - If v(t) comes from radio waves then $v(t) = A\cos(\omega t)$ and the circuit is called a radio receiver.

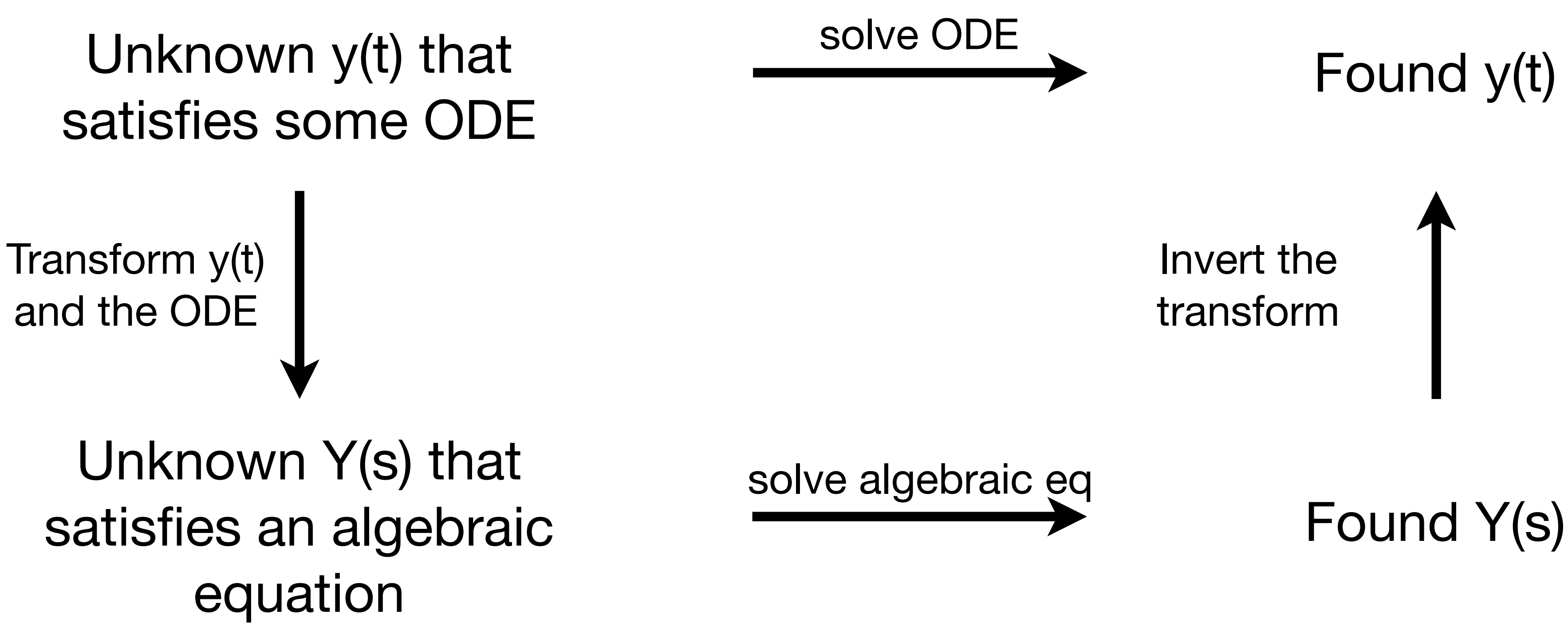
  - For $v(t) = \begin{cases} 1 & 0 < t < 10 \\ 0 & t \geq 10 \end{cases}$, the circuit has a switch that gets flipped at t=10.



Laplace transforms - intro (6.1)

- Instead of not-so-pretty techniques, we use Laplace transforms.

- Idea:



- Laplace transform of y(t): $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t)\ dt$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = 3$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} 3 \; dt$$

$$= -\frac{3}{s} e^{-st} \Big|_0^\infty$$

$$= \lim_{A \to \infty} -\frac{3}{s} e^{-st} \Big|_0^A$$

$$= -\frac{3}{s} \left(\lim_{A \to \infty} e^{-sA} - 1 \right)$$

$$= \frac{3}{s}$$ provided $s > 0$ and does not exist otherwise.

Laplace transforms - examples (6.1)

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