Today

- Diffusion equation examples and summary
- End-of-term info:
    - Don’t forget to complete the online teaching evaluation survey.
    - Next Thursday, two-stage review (optionally for 2/50 exam points).
    - Office hours during exams TBA but sometime Apr 15/16/27.

Nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$
$$u(0,t) = 9$$
$$u(2,t) = 5$$
$$u(x,0) = \sin\frac{3\pi x}{2}$$

(A) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2}$

(B) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$

★ (C) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2} + 9 - 2x$

(D) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2} + 9 - 2x$

where $b_n =$?

Nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$
$$u(0,t) = 9$$
$$u(2,t) = 5$$
$$u(x,0) = \sin\frac{3\pi x}{2}$$

(A) $b_n = \int_0^2 \sin\frac{3\pi x}{2} \cos\frac{n\pi x}{2}\, dx$

(B) $b_n = \int_0^2 \sin\frac{3\pi x}{2} \sin\frac{n\pi x}{2}\, dx$

★ (C) $b_n = \int_0^2 \left(\sin\frac{3\pi x}{2} - 9 + 2x\right) \sin\frac{n\pi x}{2}\, dx$

(D) $b_n = \int_0^2 \left(\sin\frac{3\pi x}{2} + 9 - 2x\right) \sin\frac{n\pi x}{2}\, dx$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin\frac{n\pi x}{2} + 9 - 2x$$

Nonhomogeneous boundary conditions

- How would you solve this one?

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = -2$$

$$u(x,0) = \cos\frac{3\pi x}{2}$$

For you to think about... (we can come back to this if we have time later today)

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$u(0,t) = 0$$

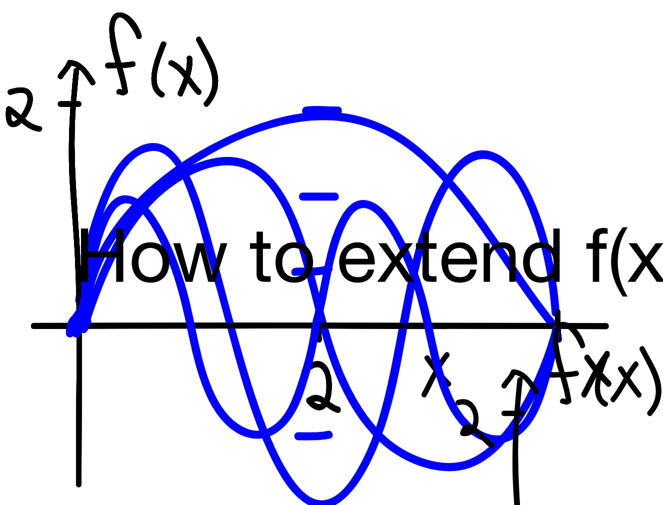
$$\left.\frac{du}{dx}\right|_{x=2} = 0$$

$$u(x,0) = x$$

Use sines? cosines?

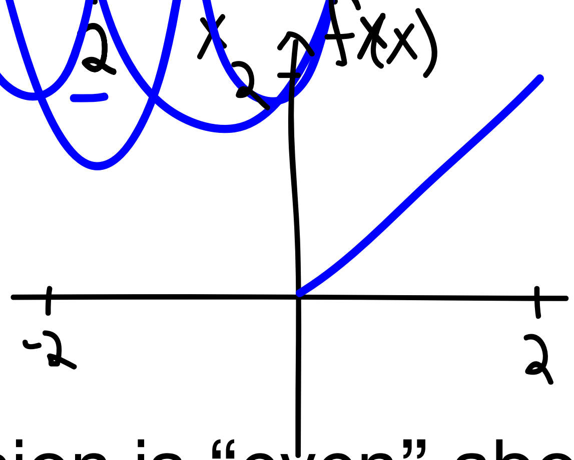
Should be zero at x=0 so definitely sine functions.

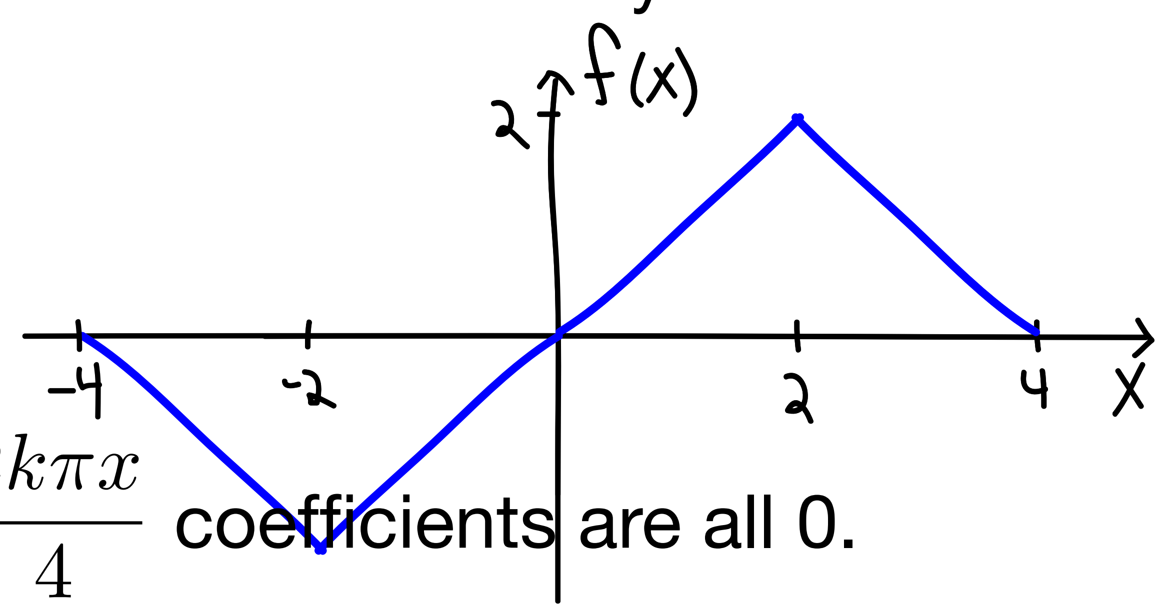
Zero slope at x=2 so extend to x=4 and choose periods to get the slope right.



$$\sin\frac{n\pi x}{4}: \quad \sin\frac{\pi x}{4} \quad \sin\frac{2\pi x}{4} \quad \sin\frac{3\pi x}{4} \quad \sin\frac{4\pi x}{4}$$

How to extend f(x) so that its Fourier sine series has only odd values of n?





Extension is "even" about x=2 so $\sin\frac{2k\pi x}{4}$ coefficients are all 0.

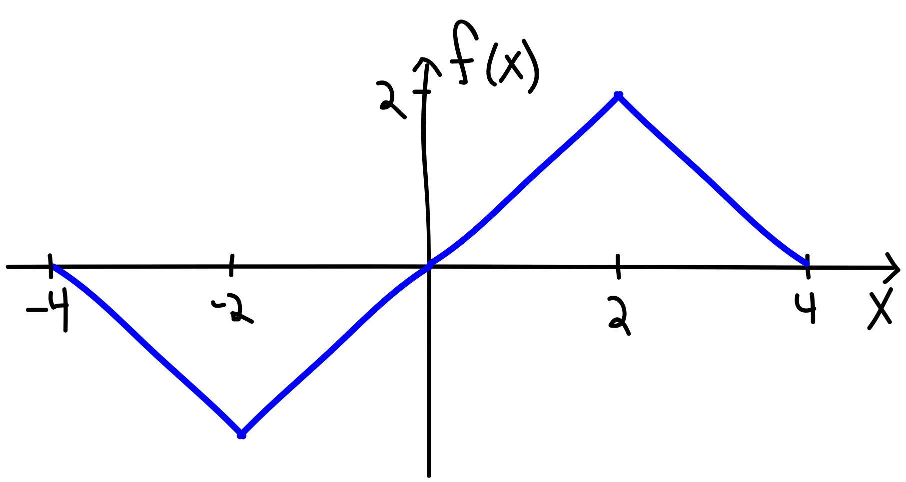
Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$u(0,t) = 0$$

$$\left.\frac{du}{dx}\right|_{x=2} = 0$$

$$u(x,0) = x$$



$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4\frac{n^2\pi^2}{16}t} \sin\left(\frac{n\pi x}{4}\right)$$

$$x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

$$b_n = \ldots = \frac{16}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & n = 1, 5, 9\ldots \\ 0 & n = 2, 6, 10\ldots \\ -1 & n = 3, 7, 11\ldots \\ 0 & n = 4, 8, 12\ldots \end{cases}$$

Optionally: $n = 2k - 1$

$$x = \sum_{k=1}^{\infty} b_{2k-1} \sin\left(\frac{(2k-1)\pi x}{4}\right)$$

$$b_{2k-1} = \frac{16}{(2k-1)^2\pi^2}(-1)^{k+1}$$

https://www.desmos.com/calculator/aoqcibcvho

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$u(0,t) = 3$$

$$\left.\frac{du}{dx}\right|_{x=2} = 8$$

$$u(x,0) = 9x + 3$$

$$u_{ss}(x) = 3 + 8x$$

$$v(x,t) = u(x,t) - u_{ss}(x)$$

$$v(0,t) = 0$$

$$\left.\frac{dv}{dx}\right|_{x=2} = 0$$

$$v(x,0) = u(x,0) - u_{ss}(x) = x$$

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-4\frac{n^2\pi^2}{16}t} \sin\left(\frac{n\pi x}{4}\right)$$

$$b_n = \frac{16}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$u(x,t) = u_{ss}(x) + \sum_{n=1}^{\infty} b_n e^{-4\frac{n^2\pi^2}{16}t} \sin\left(\frac{n\pi x}{4}\right)$$

$$= 3 + 8x + \sum_{n=1}^{\infty} b_n e^{-4\frac{n^2\pi^2}{16}t} \sin\left(\frac{n\pi x}{4}\right)$$

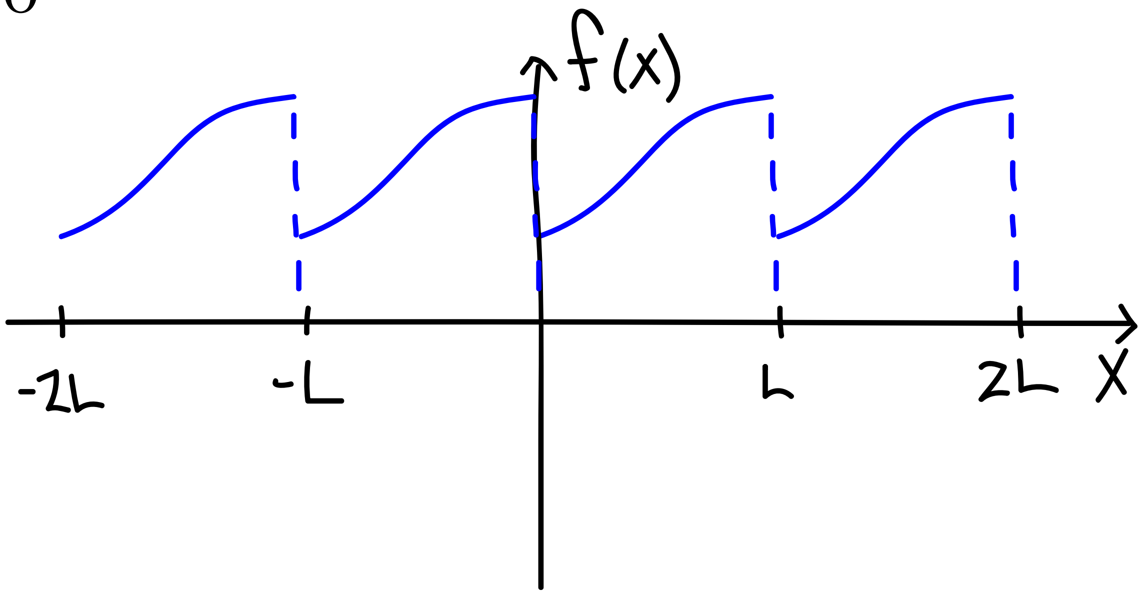
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend f(x) to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- All coefficients will be non-zero. Not particularly useful for solving the BCs.

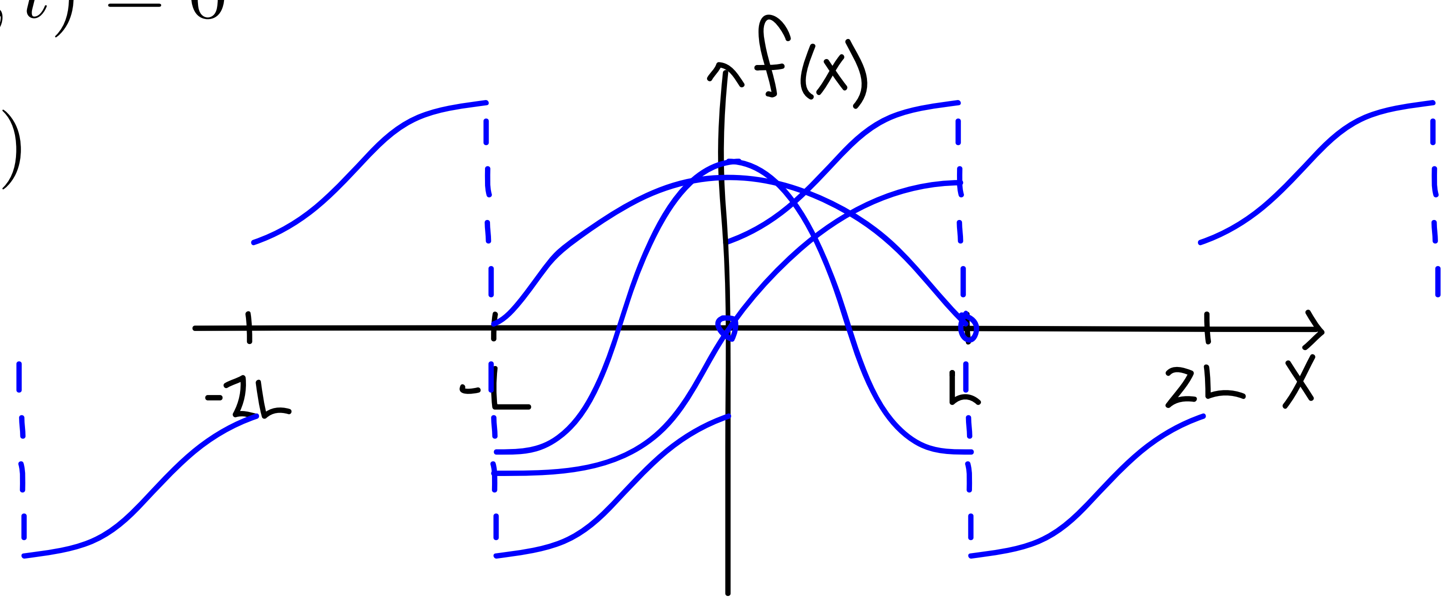
Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend to -L as an odd function and then to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- Cosine coefficients will be zero because f(x) is odd about x=0 and cosine is even. Useful for solving the Diffusion equation with Dirichlet BCs.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$u_t = Du_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = 0$$

$$u(x,0) = f(x)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x\right)\sin\frac{n\pi x}{L}\,dx$$

- Adding the linear function to the usual solution to the Dirichlet problem ensures that the BCs are satisfied without changing the fact that it satisfies the PDE.

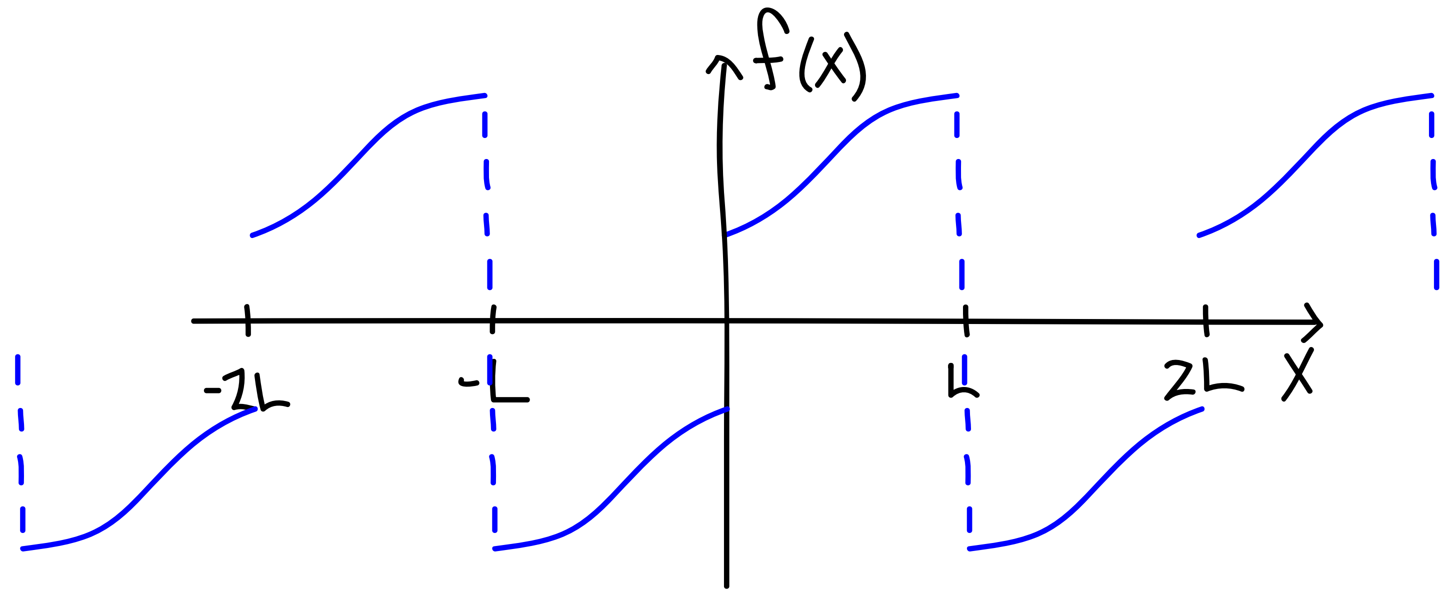
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

$$u(x,0) = f(x)$$



Odd symmetry about x=0

Odd symmetry about x=L

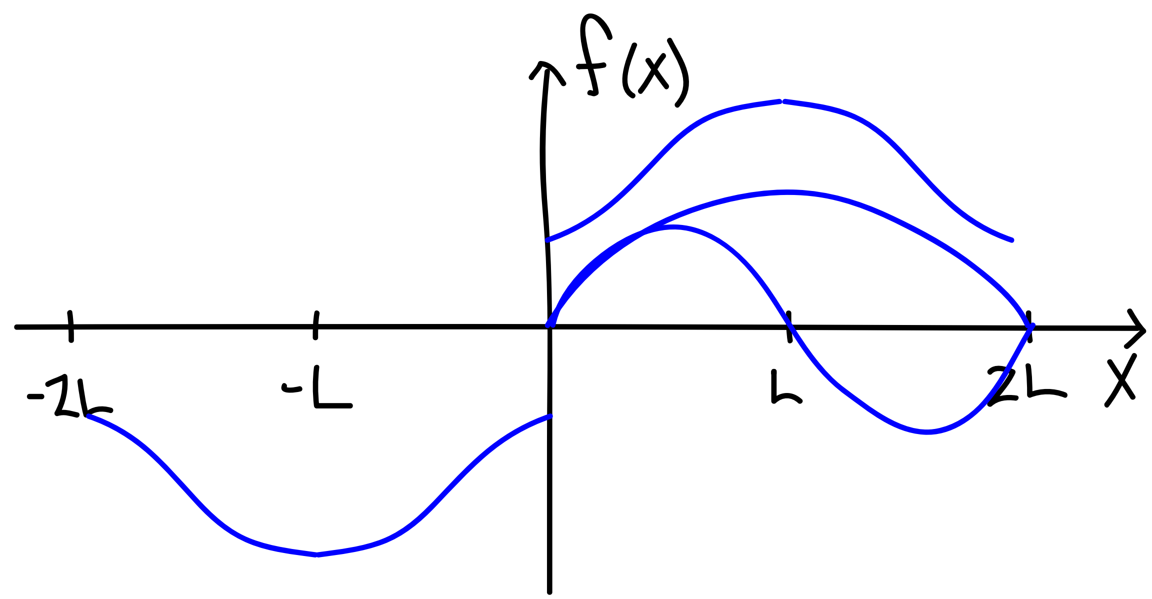
Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$u(0,t) = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

$$u(x,0) = f(x)$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2L}$$

$$b_n = \frac{1}{2L}\int_{-2L}^{2L} f(x)\sin\frac{n\pi x}{2L}\,dx$$

$$= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{2L}\,dx \quad = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{2L}\,dx$$

(for n odd)

$$b_n = \begin{cases} 0 & \text{for } n \text{ even,} \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx & \text{for } n \text{ odd.} \end{cases}$$

Review of solutions to the Diffusion Equation

- Diffusion equation with
  - Homogeneous
    - Pure Dirichlet BCs (u=0) --> use sin(nπx / L).
    - Pure Neumann BCs (u_x=0) --> use cos(nπx / L).
    - Mixed Dirichlet/Neumann --> use sin(nπx / 2L).
    - Mixed Neumann/Dirichlet --> use cos(nπx / 2L).
  - Nonhomogeneous
    - Find steady state, subtract from f(x), find FS as above, add back steady state.