Today

- Shapes of solutions for distinct eigenvalues case.
- General solution for complex eigenvalues case.
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Shapes of solution curves in the phase plane

- When matrix A has distinct eigenvalues, the general solution to **x**'=A**x** is

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2}$$

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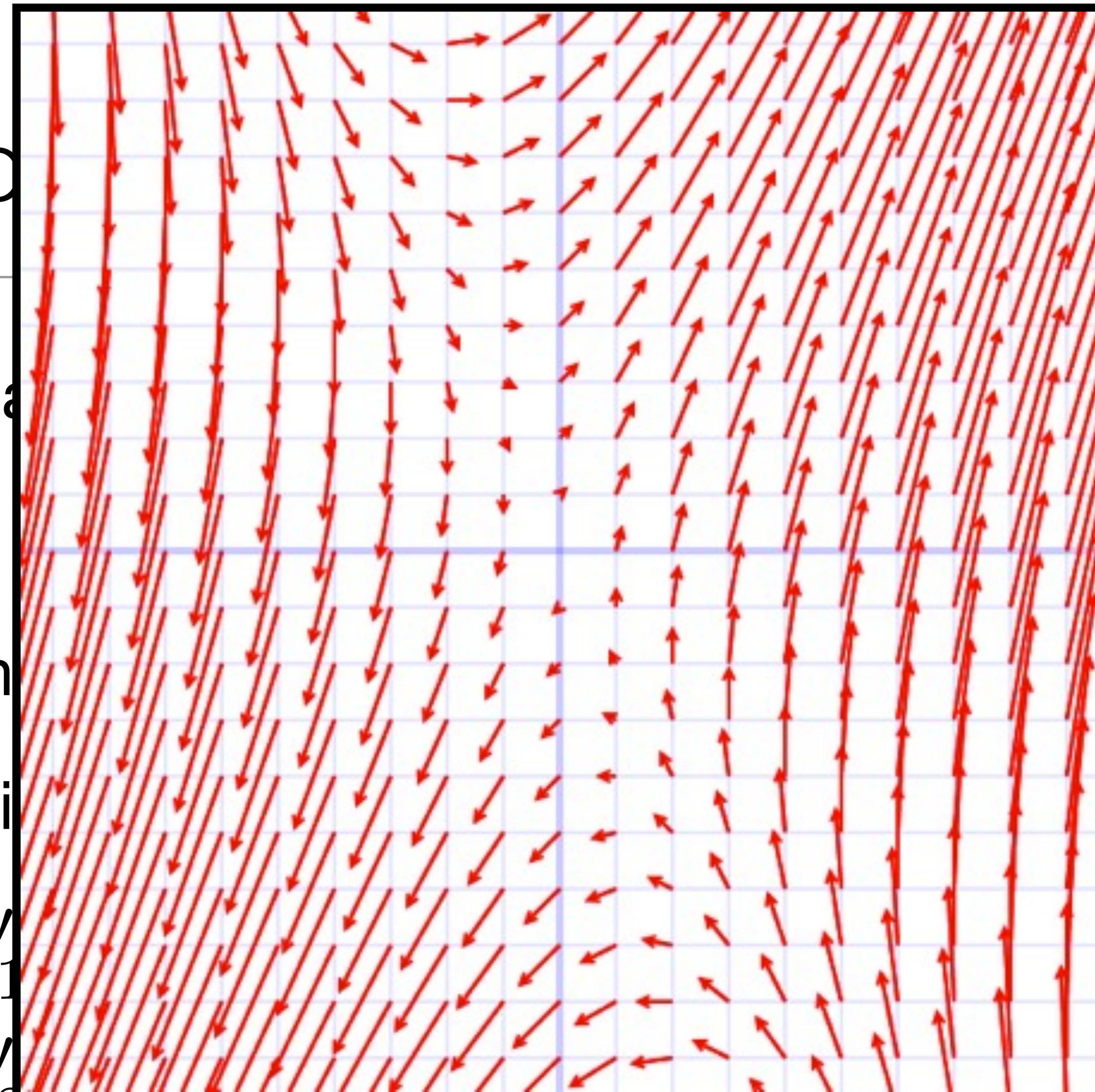

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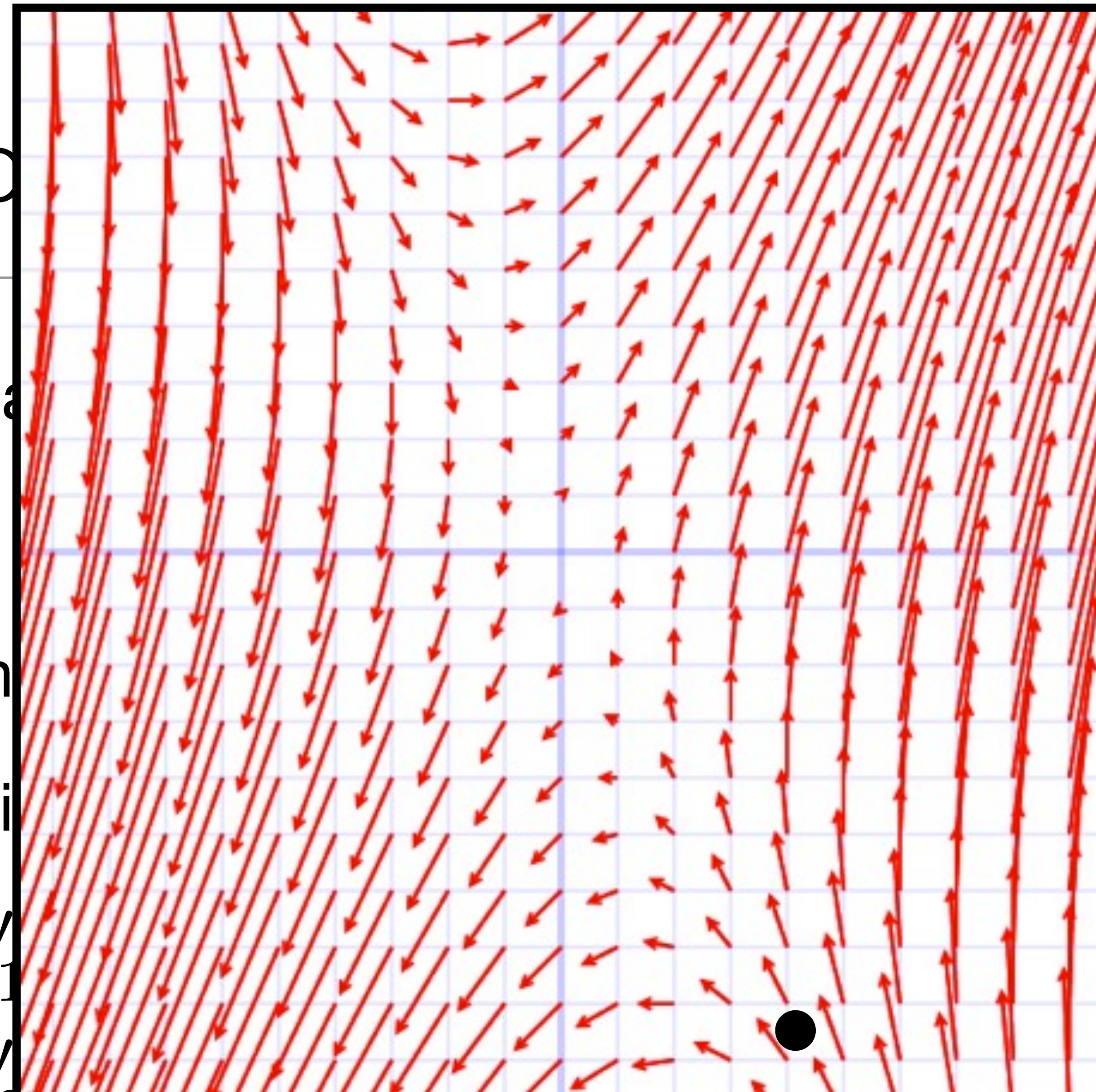

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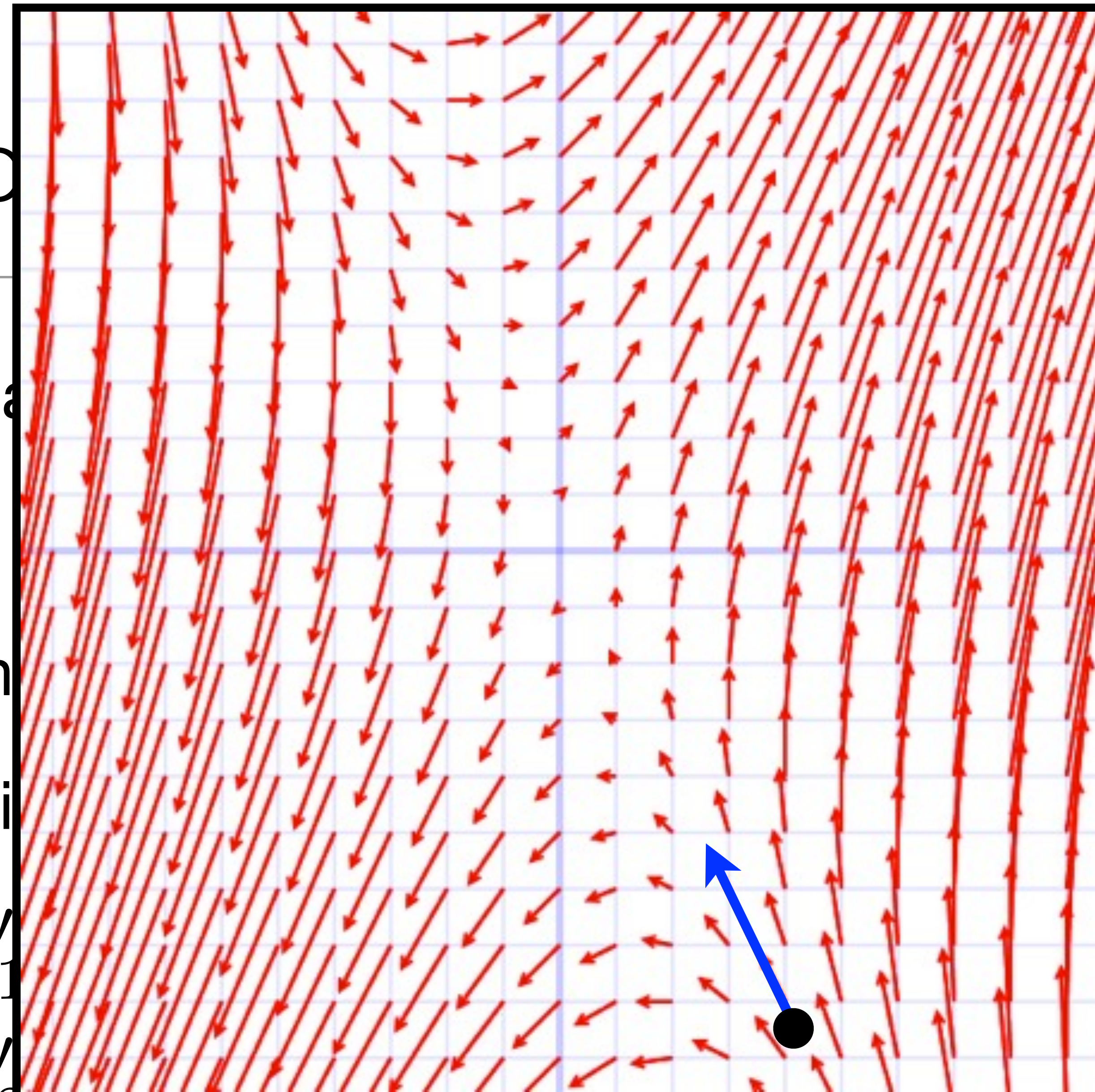

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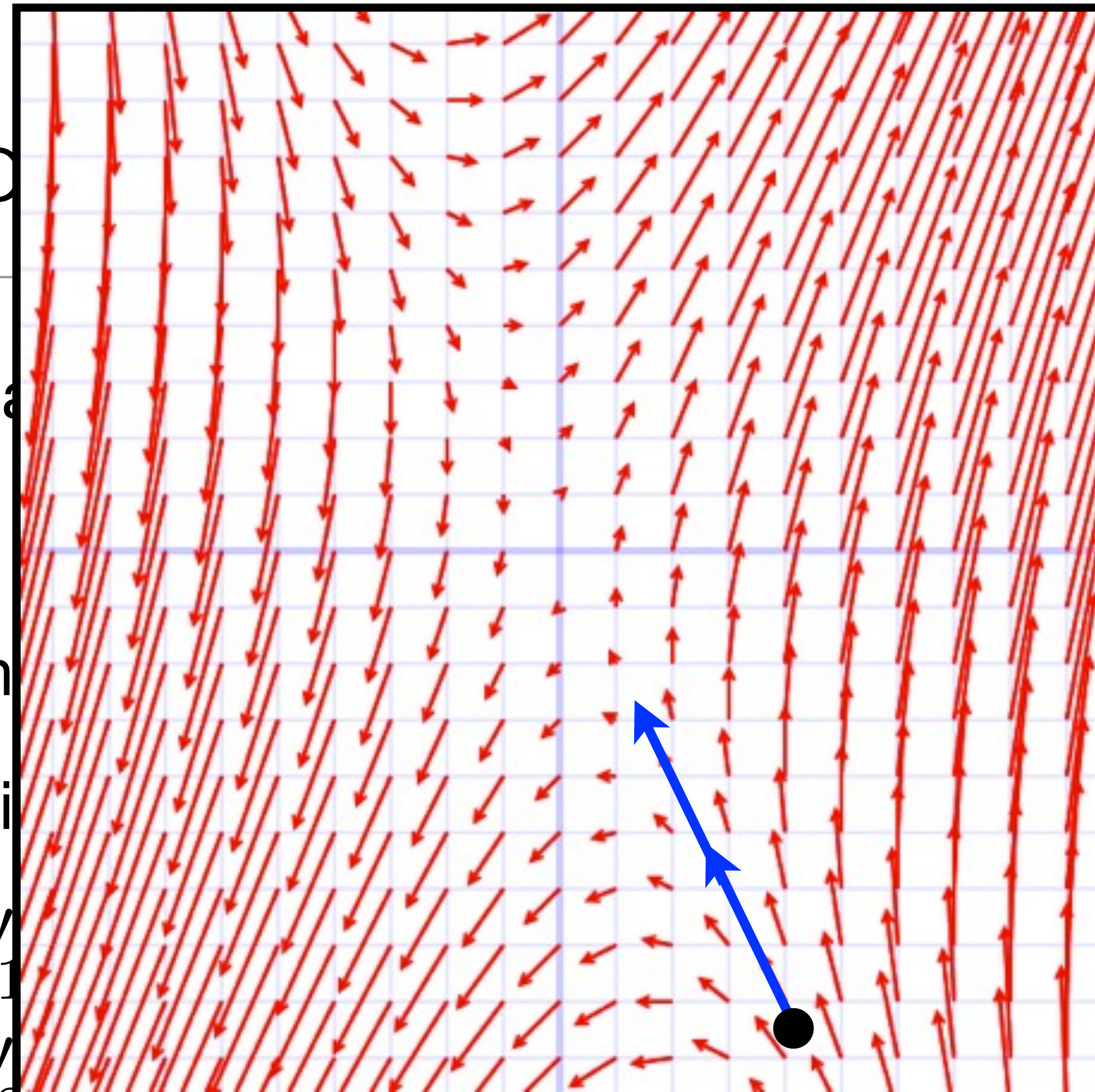

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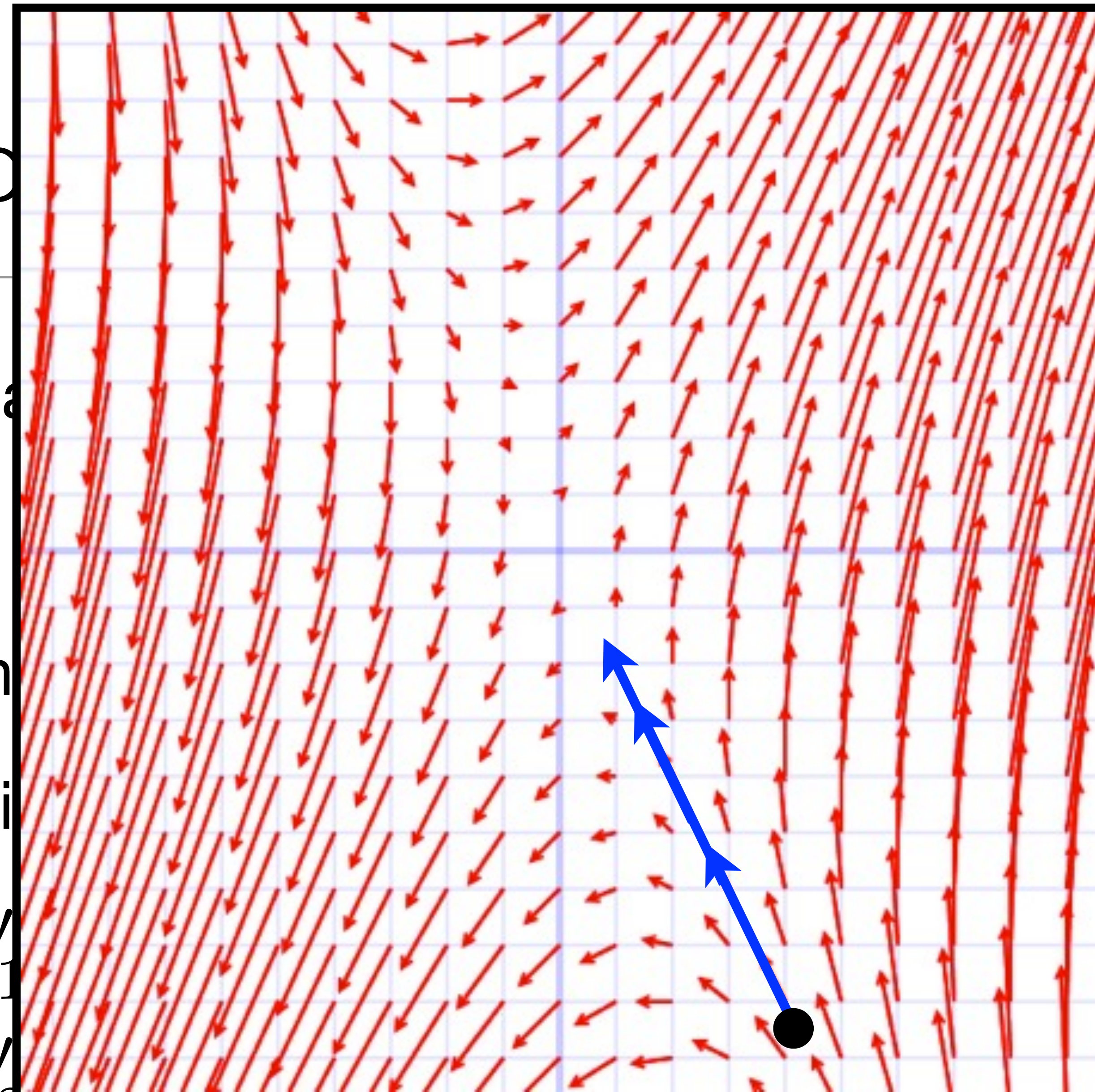

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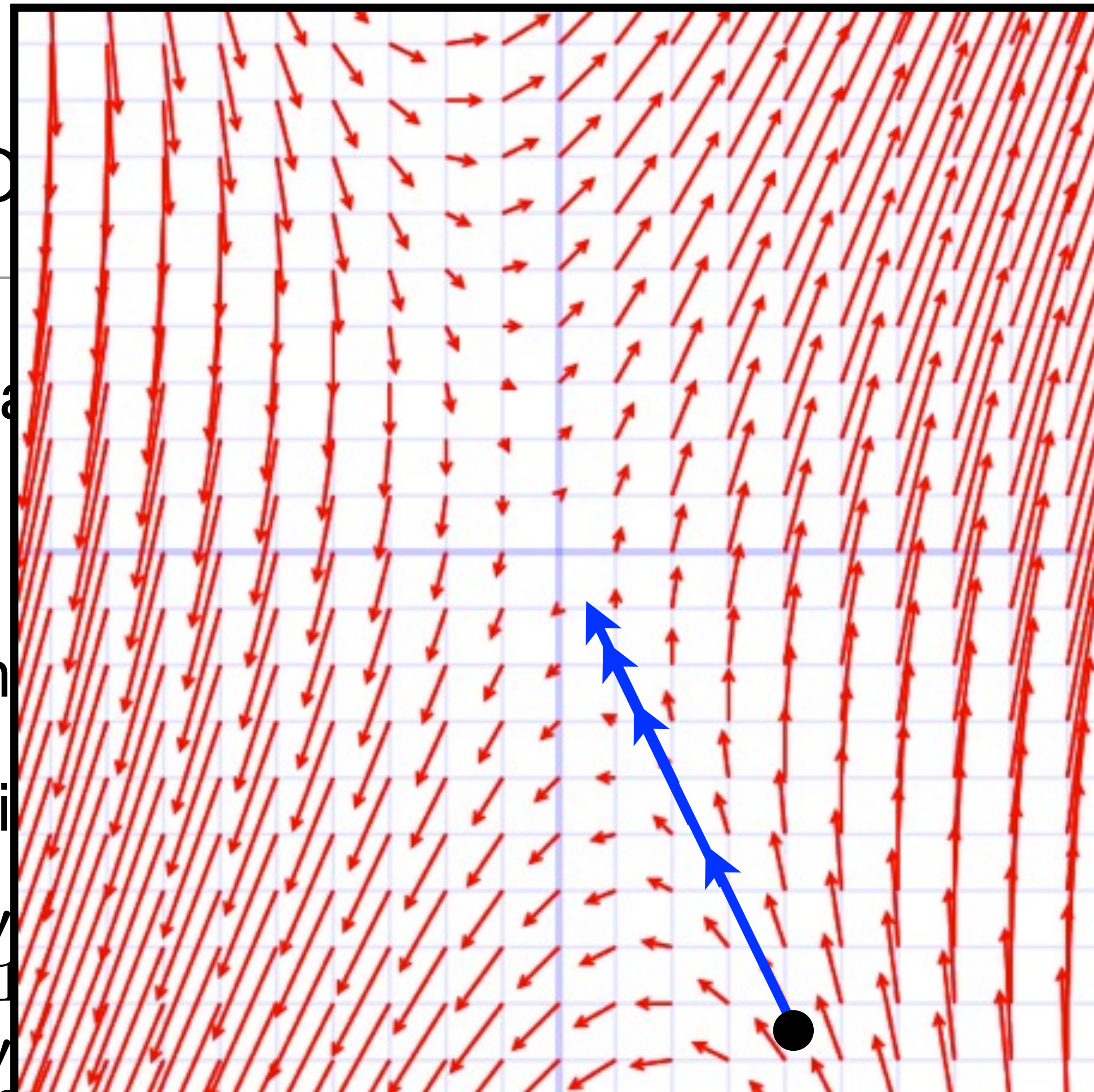

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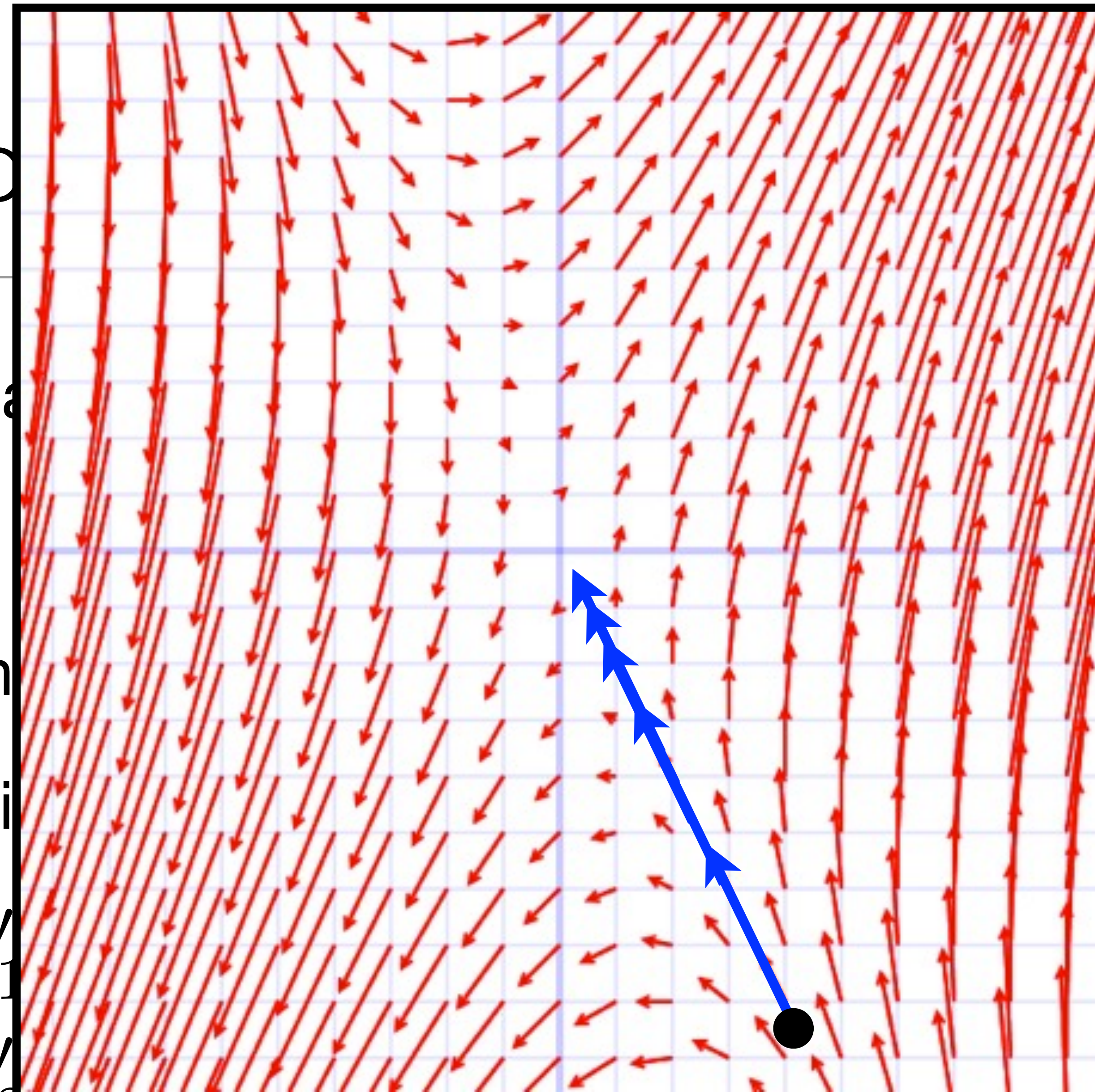

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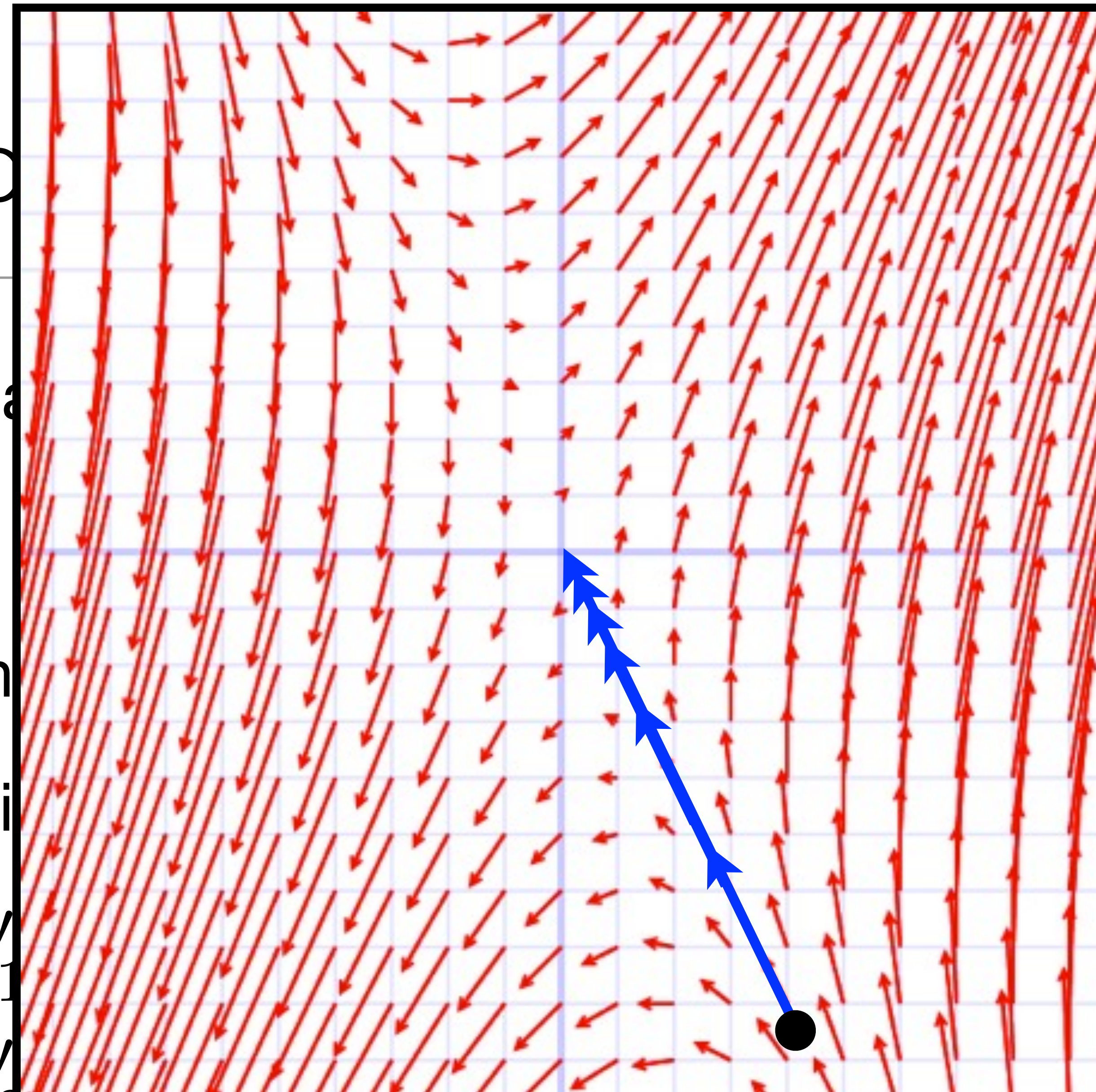

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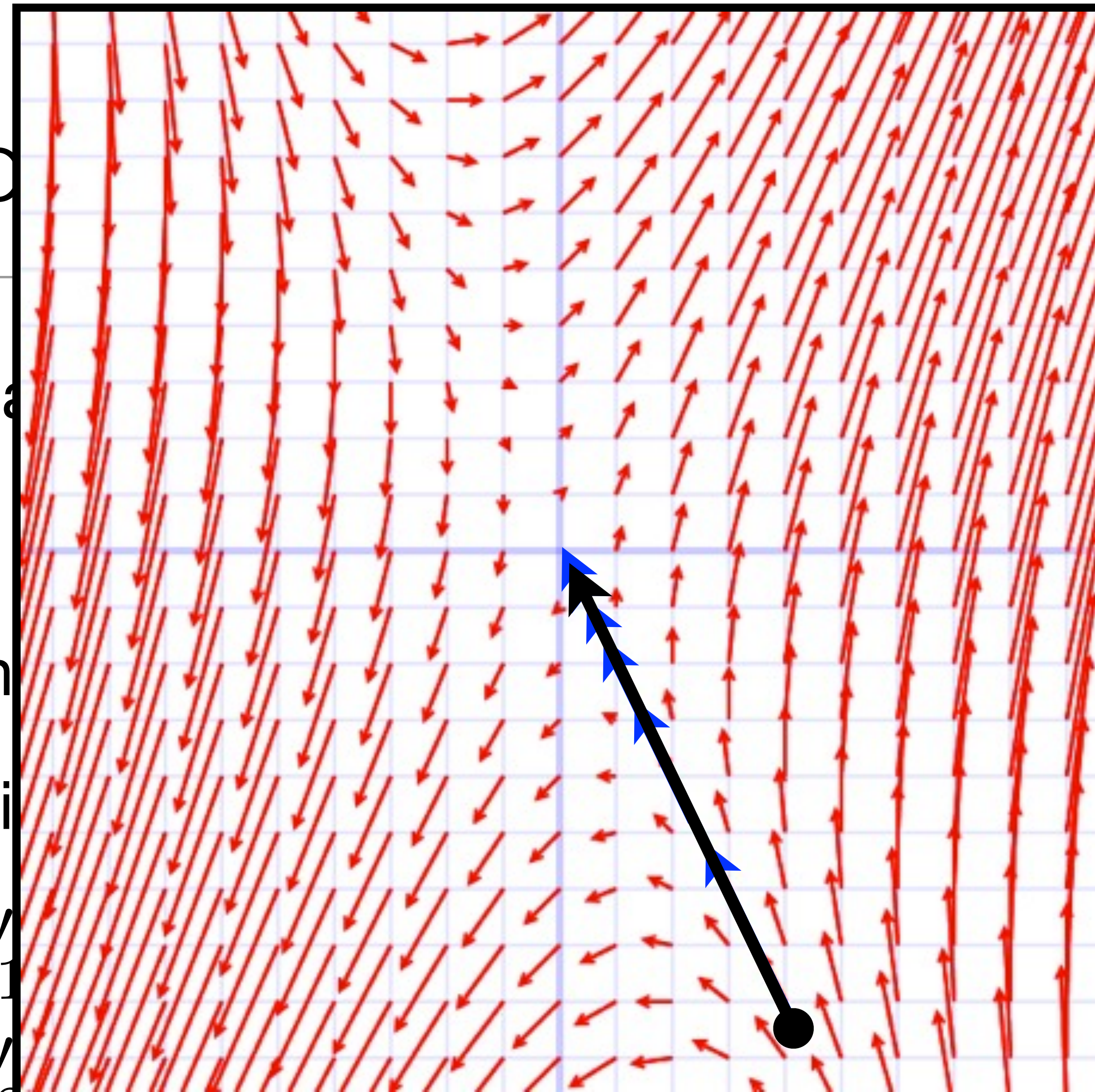

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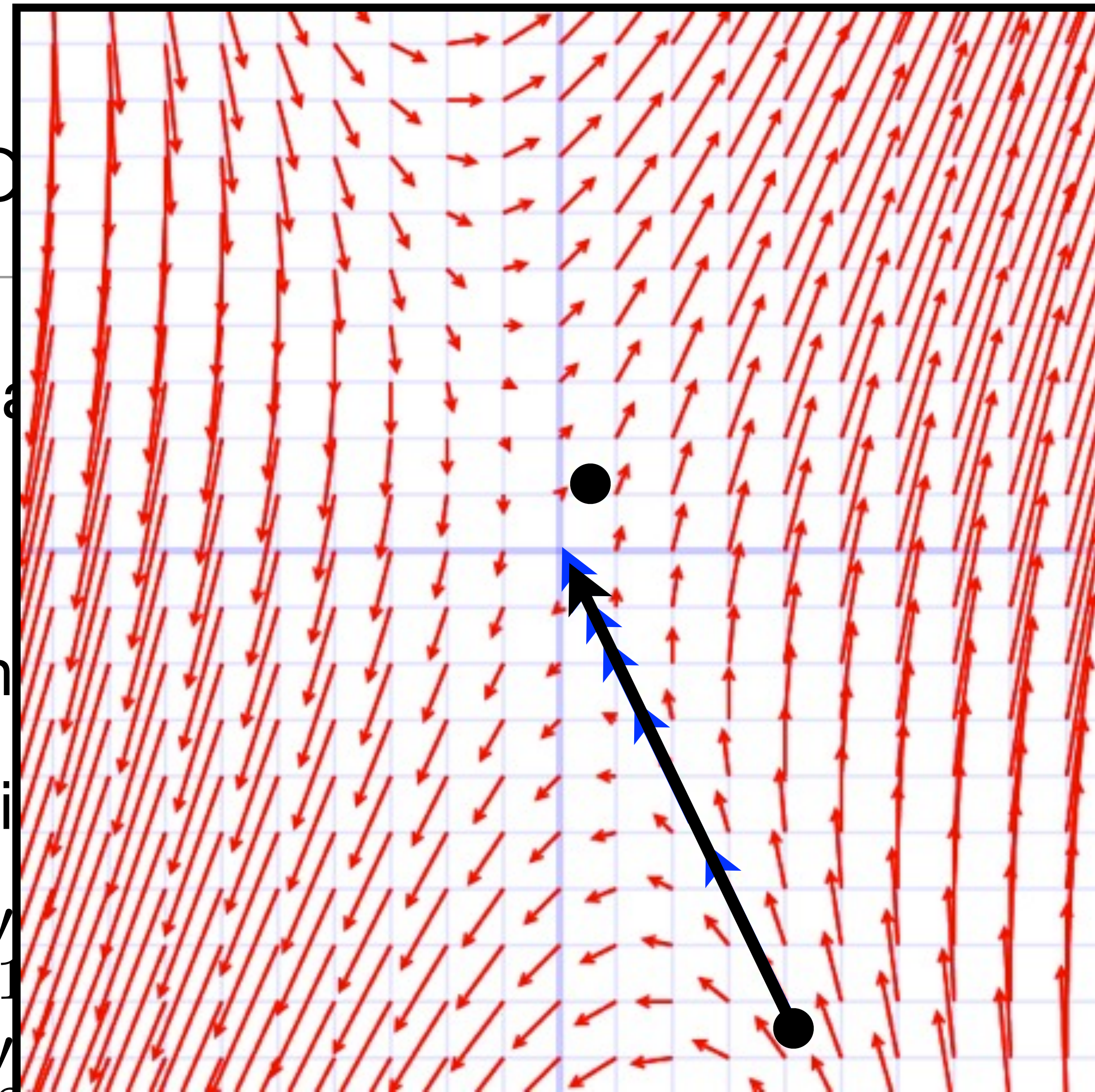

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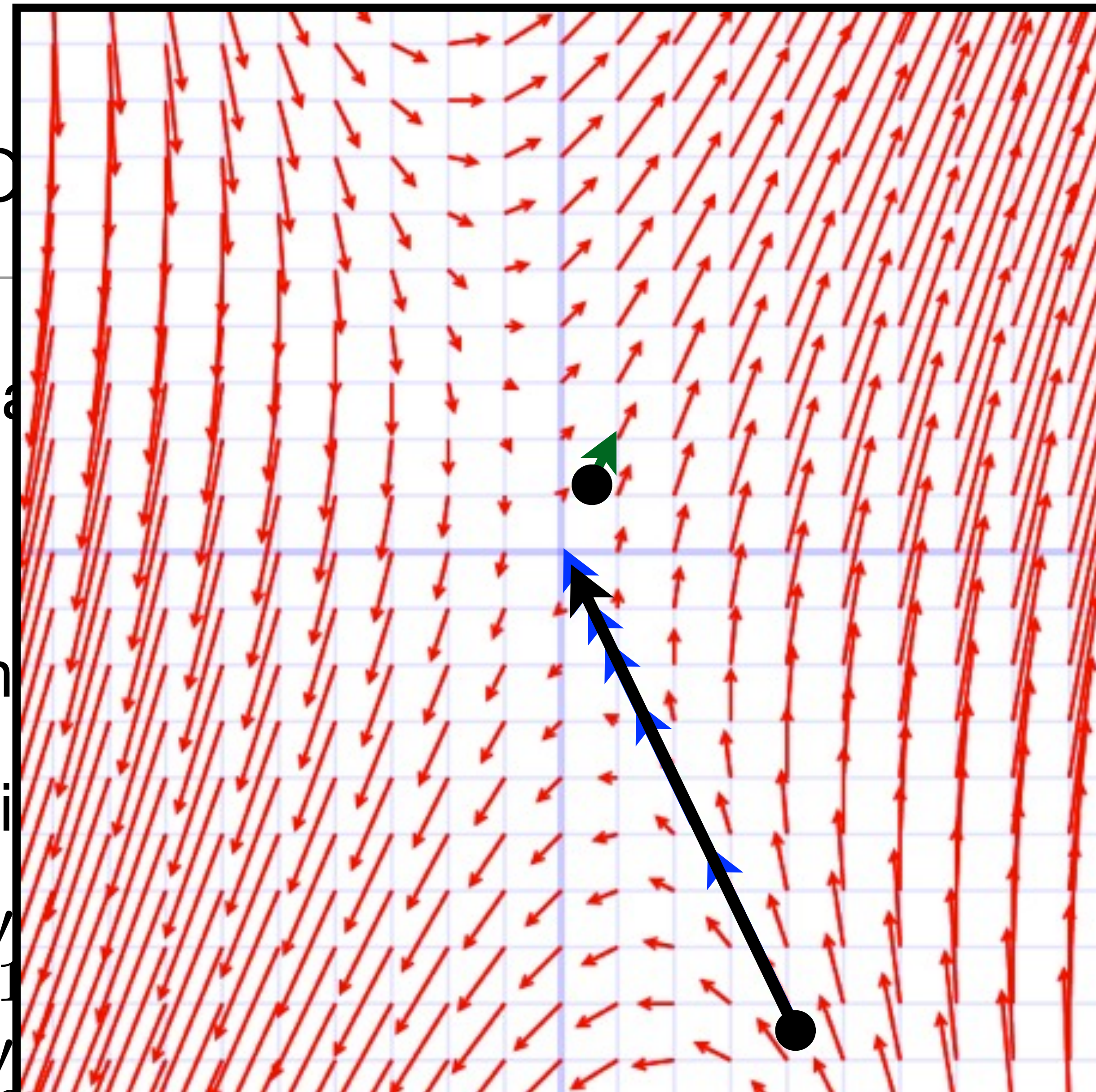

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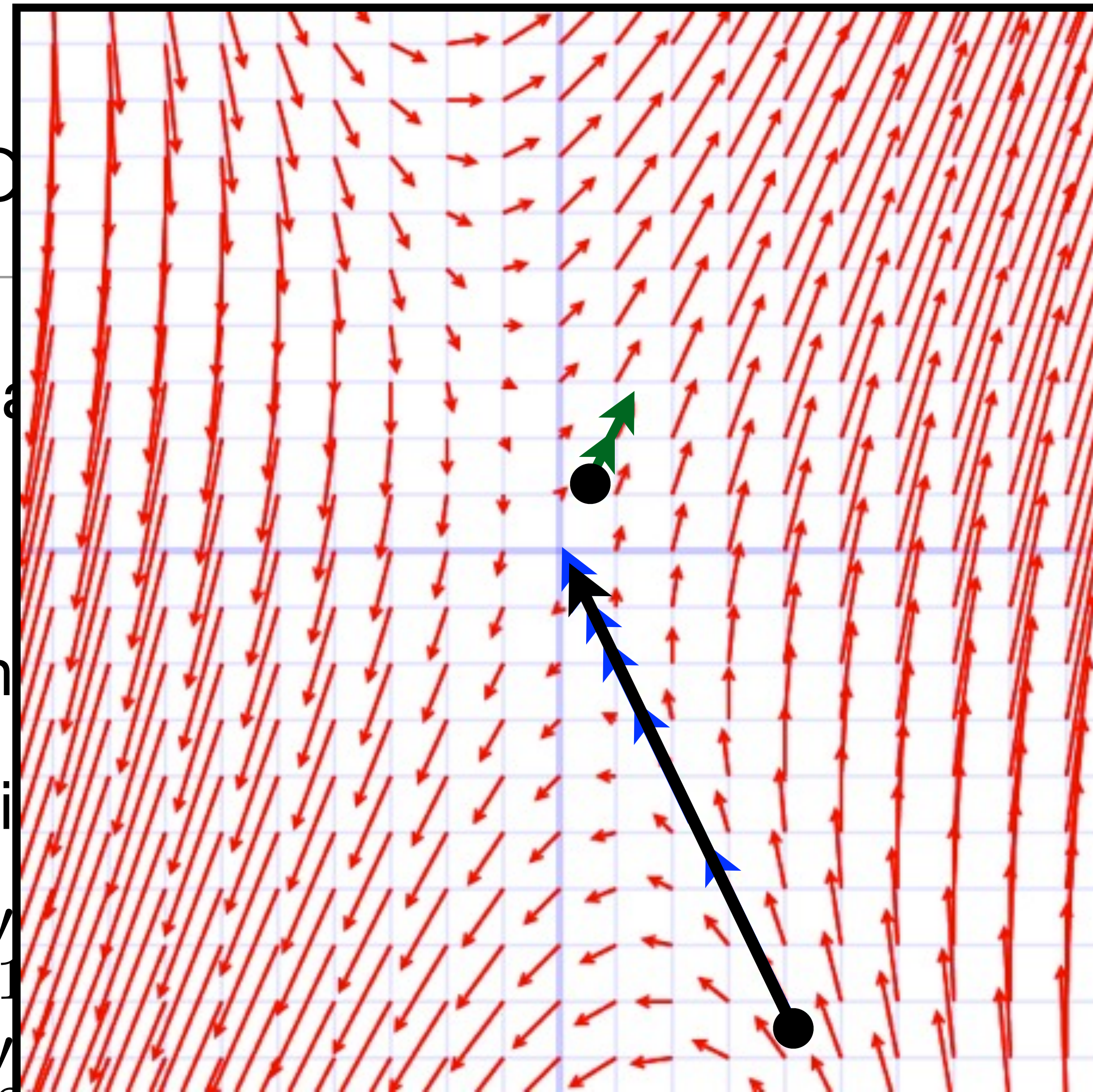
Shapes of so  e plane

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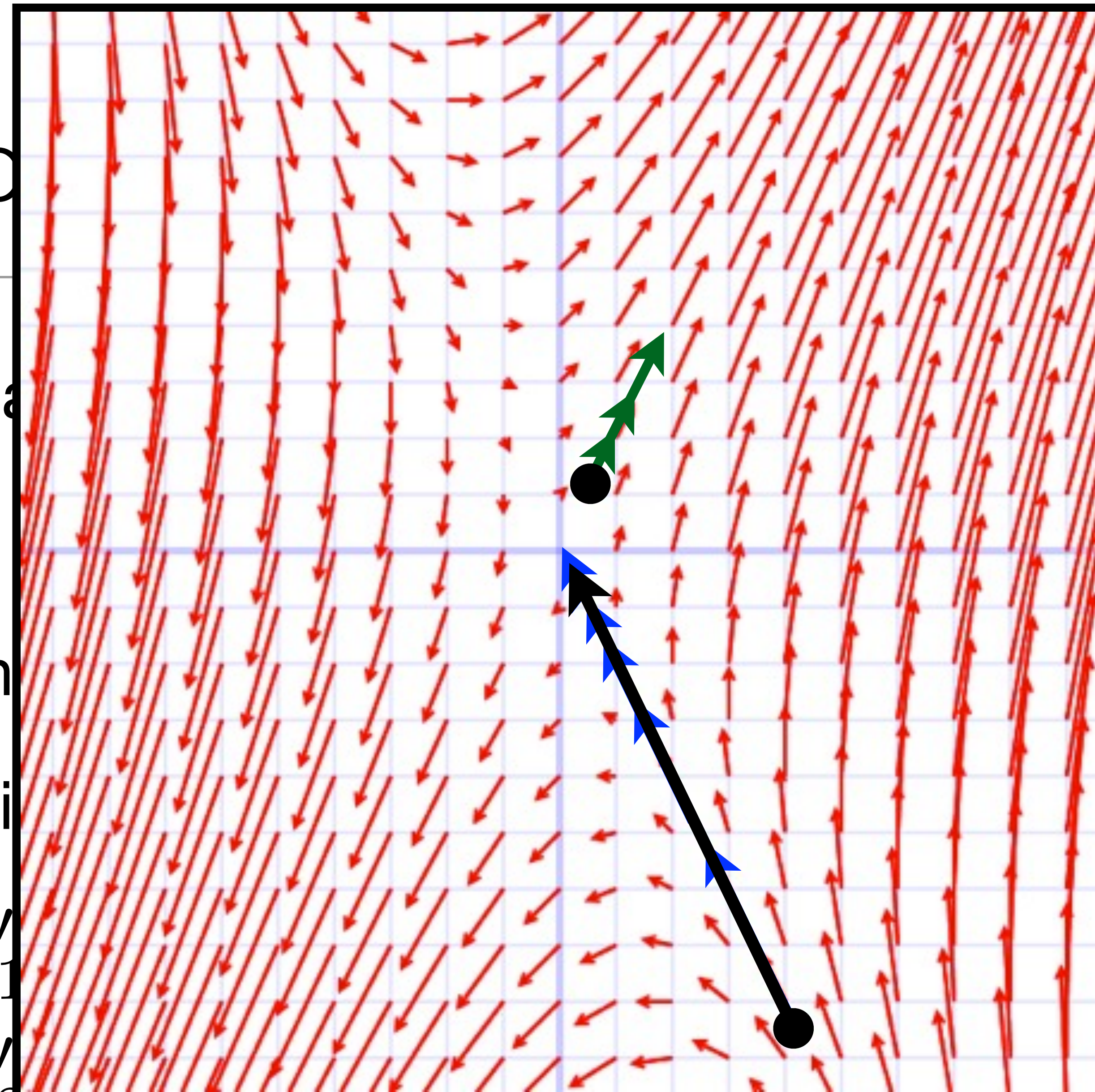

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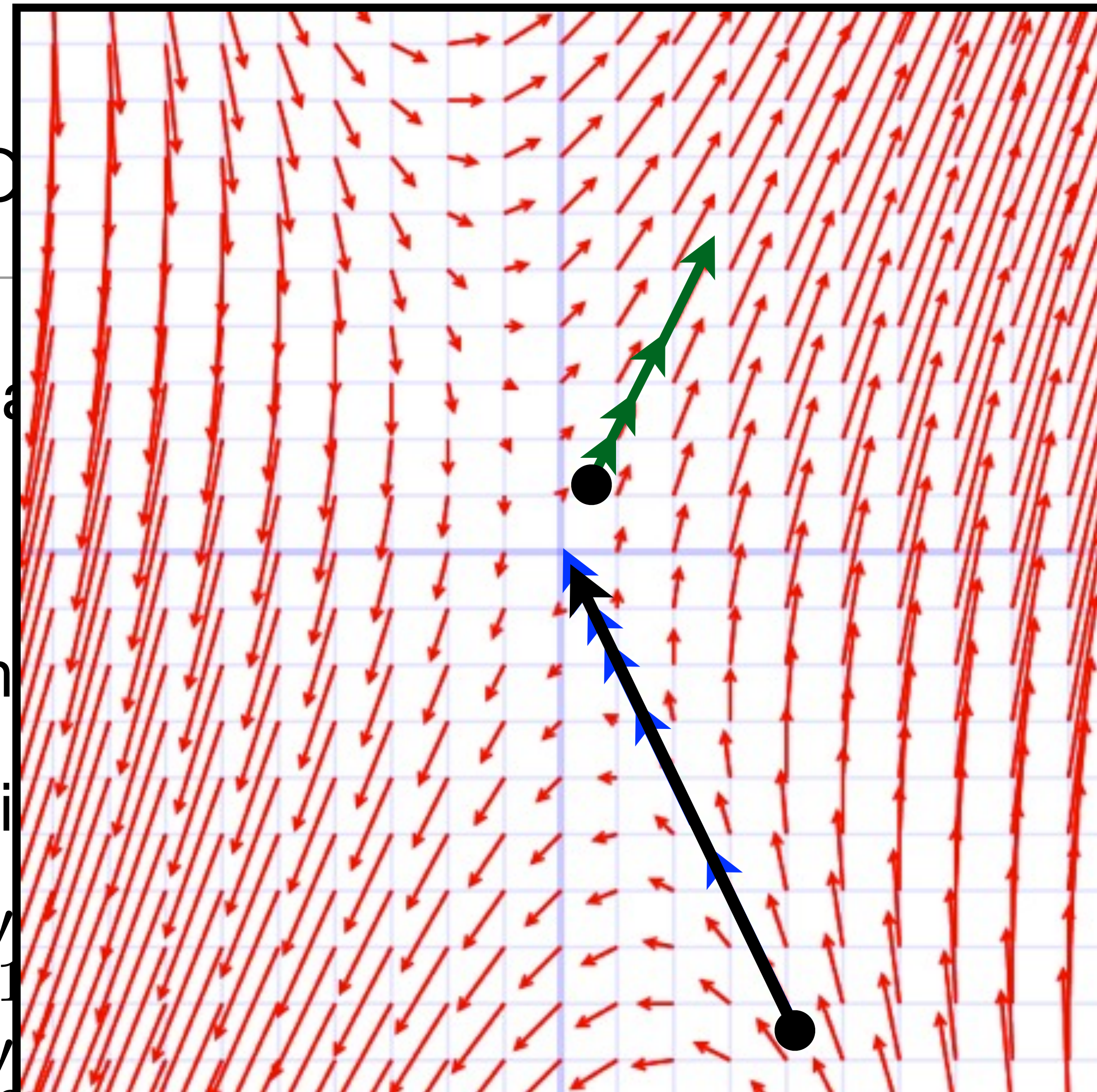

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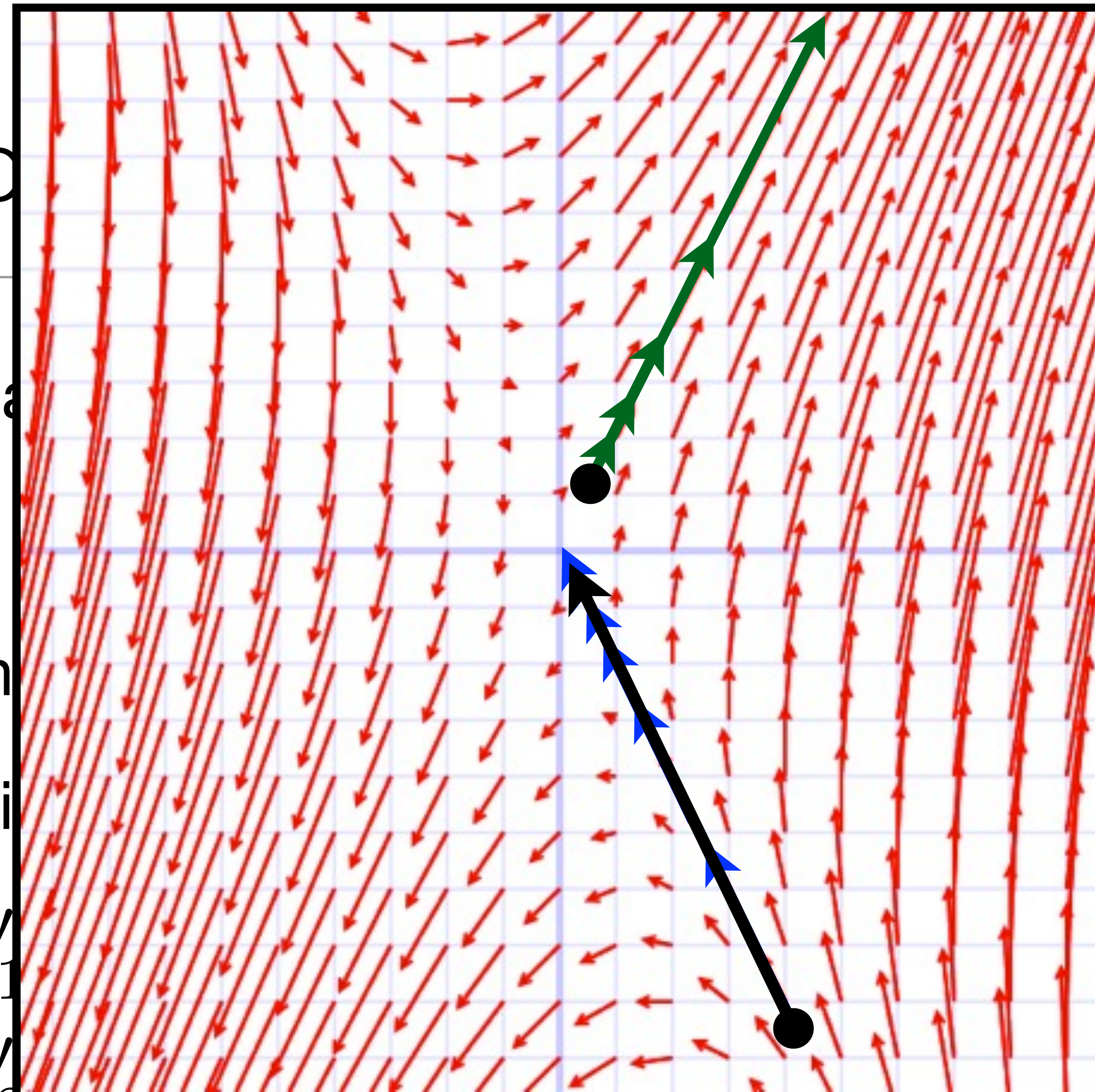

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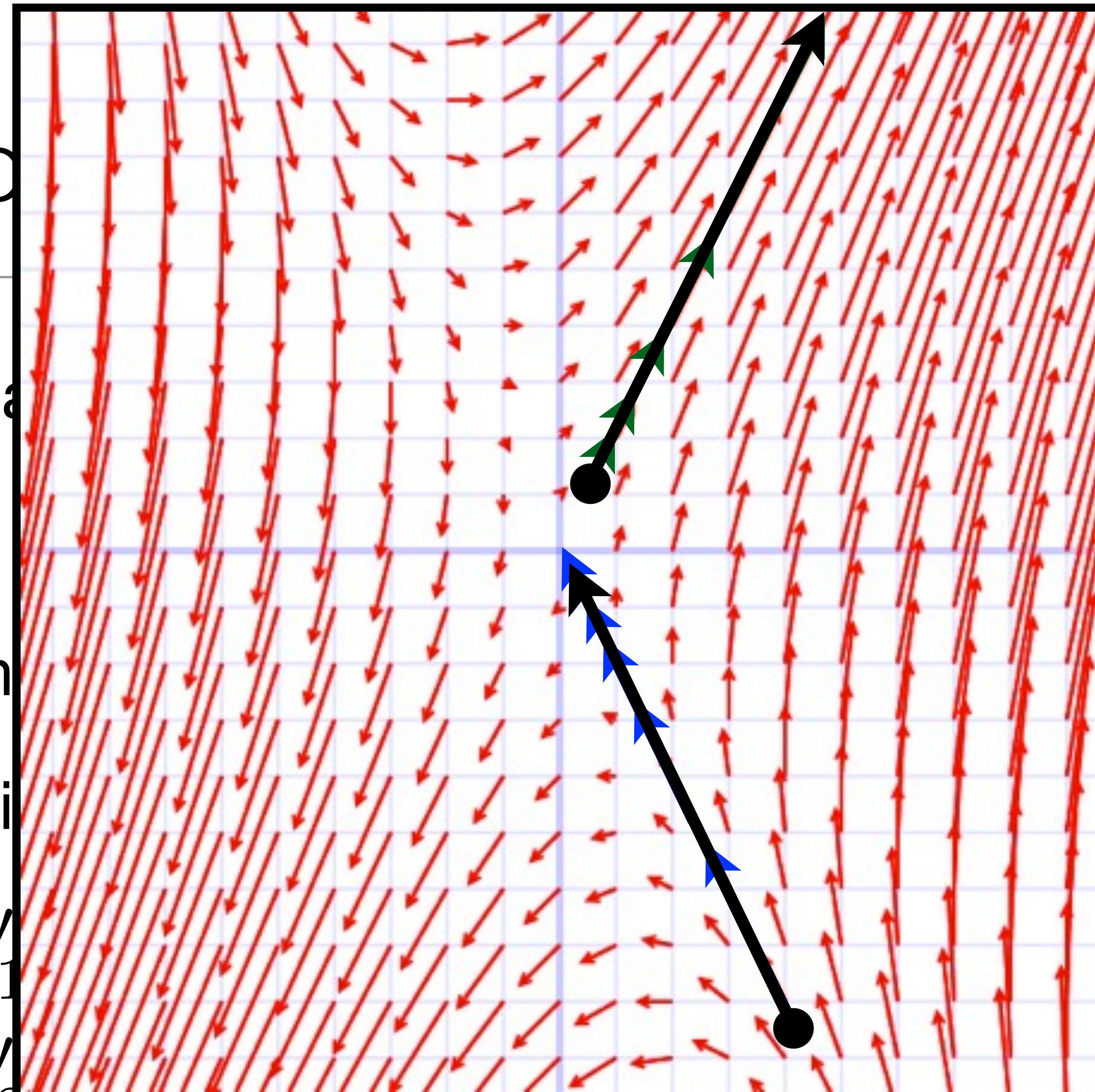

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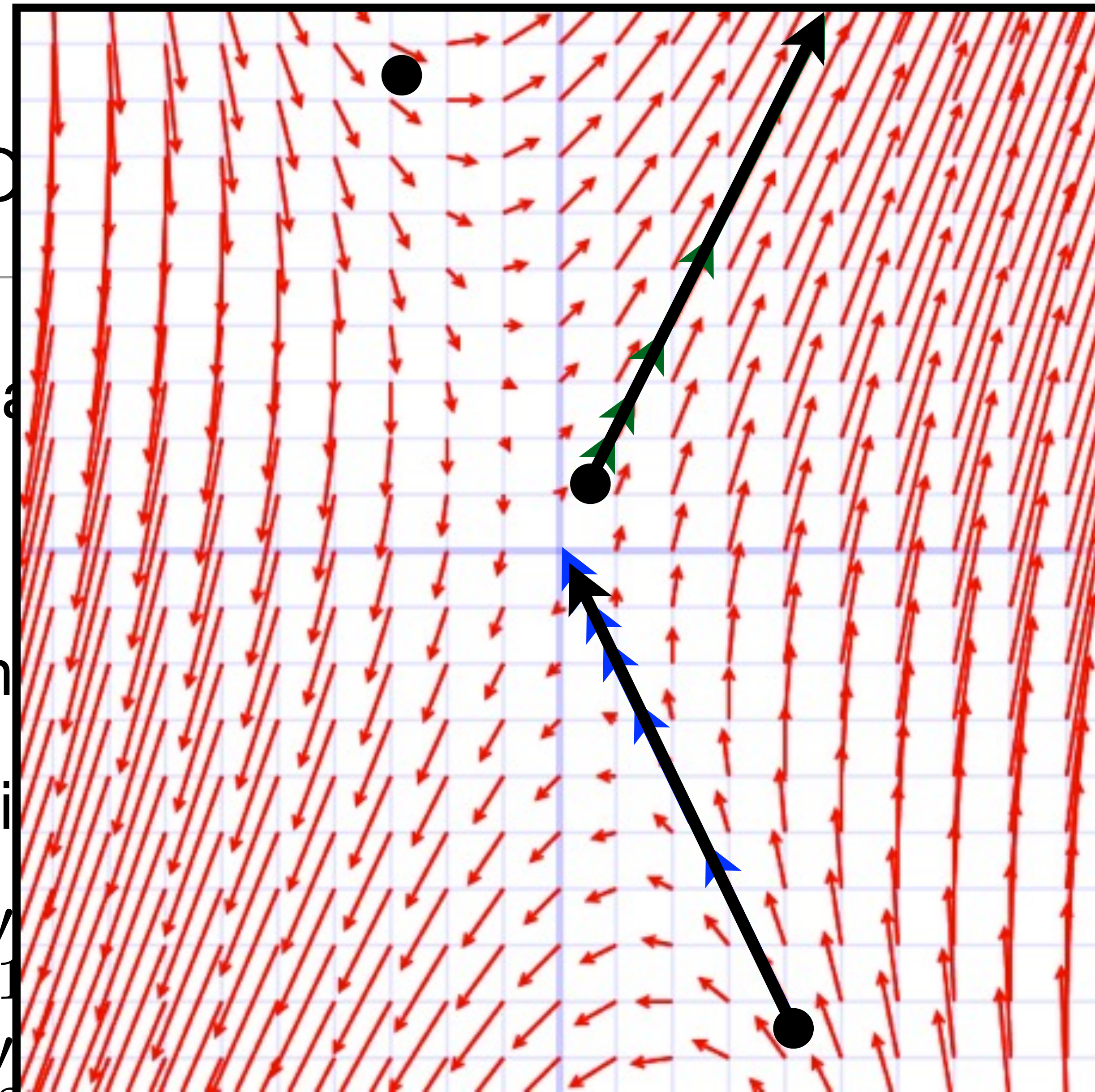

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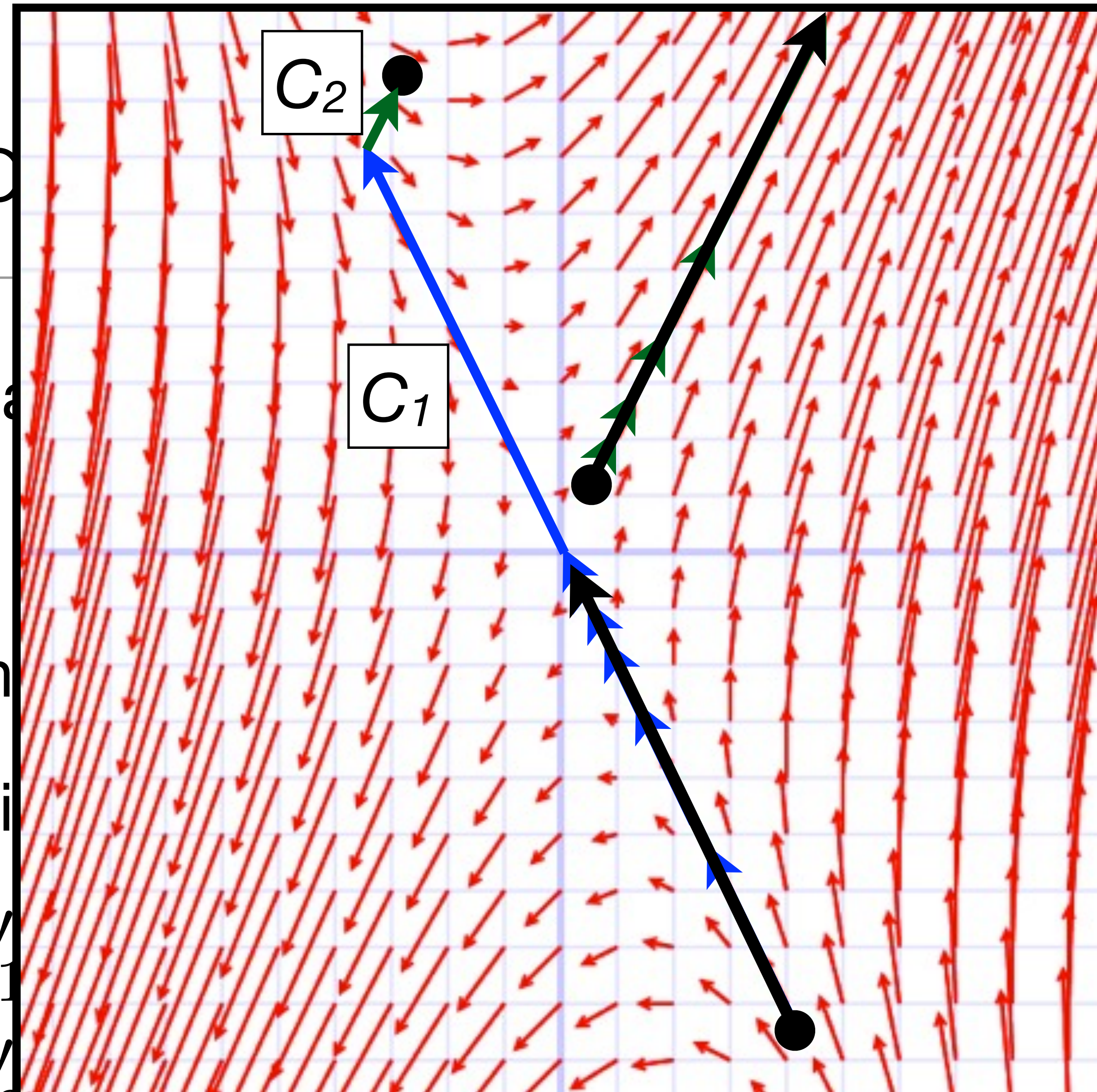

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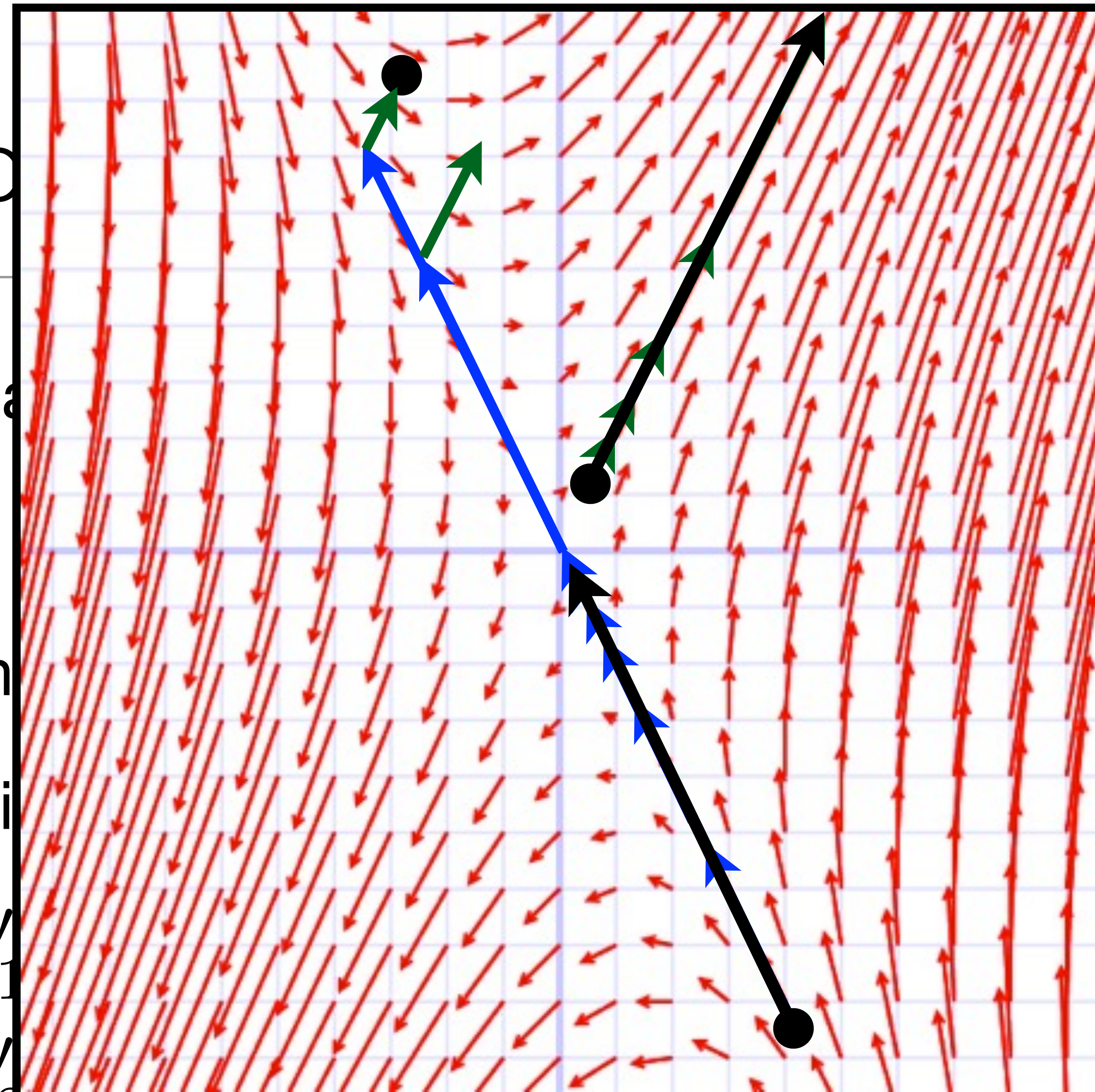

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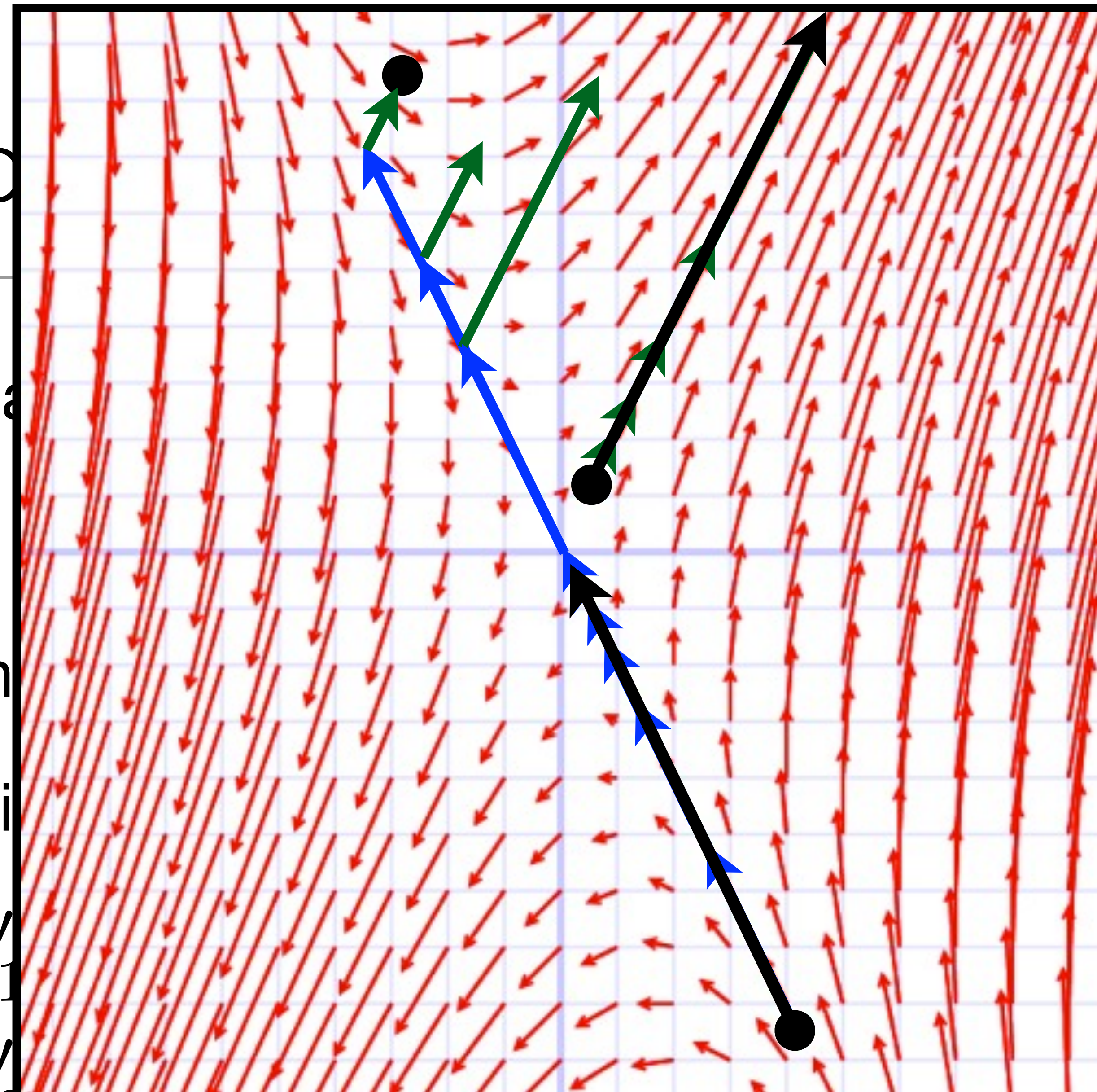

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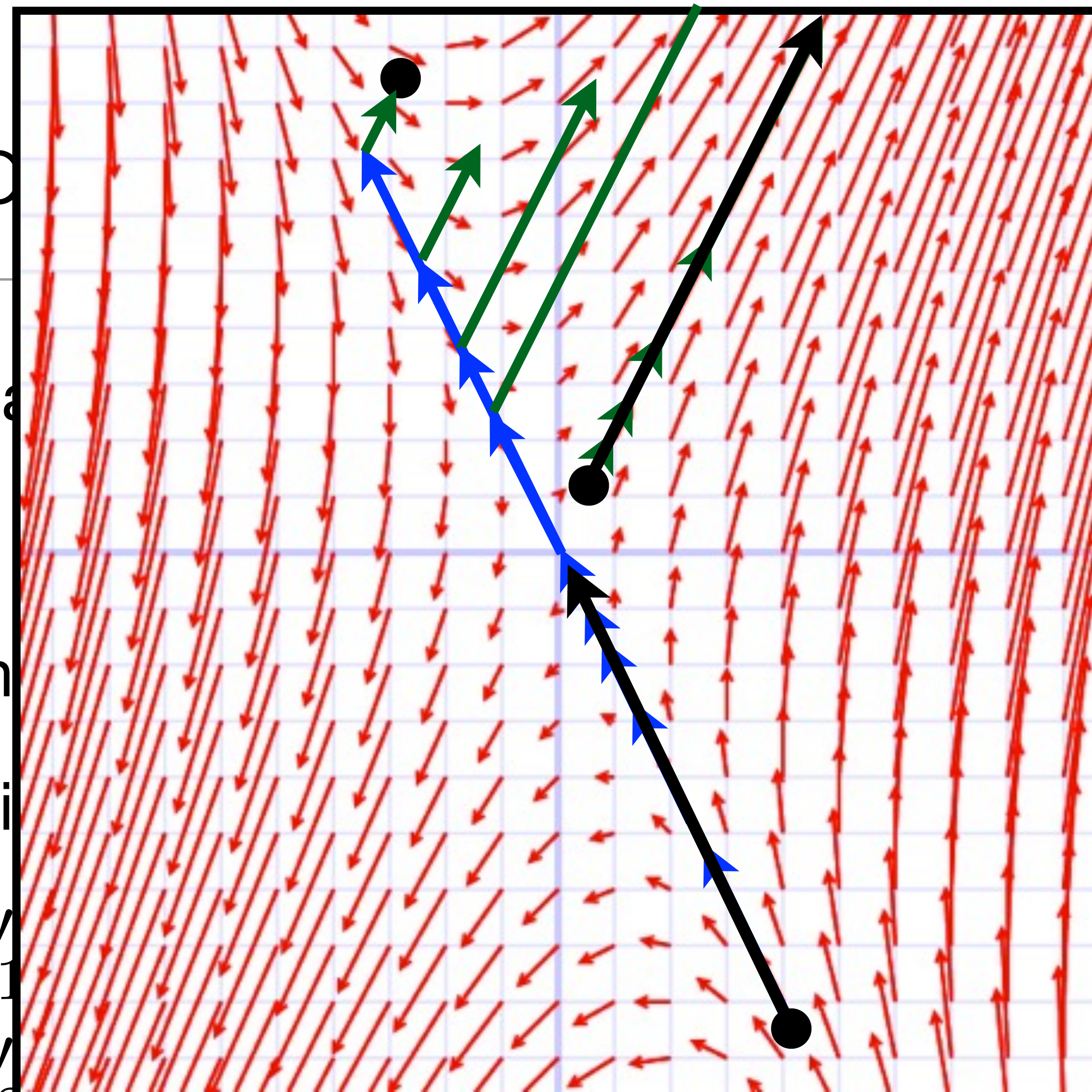

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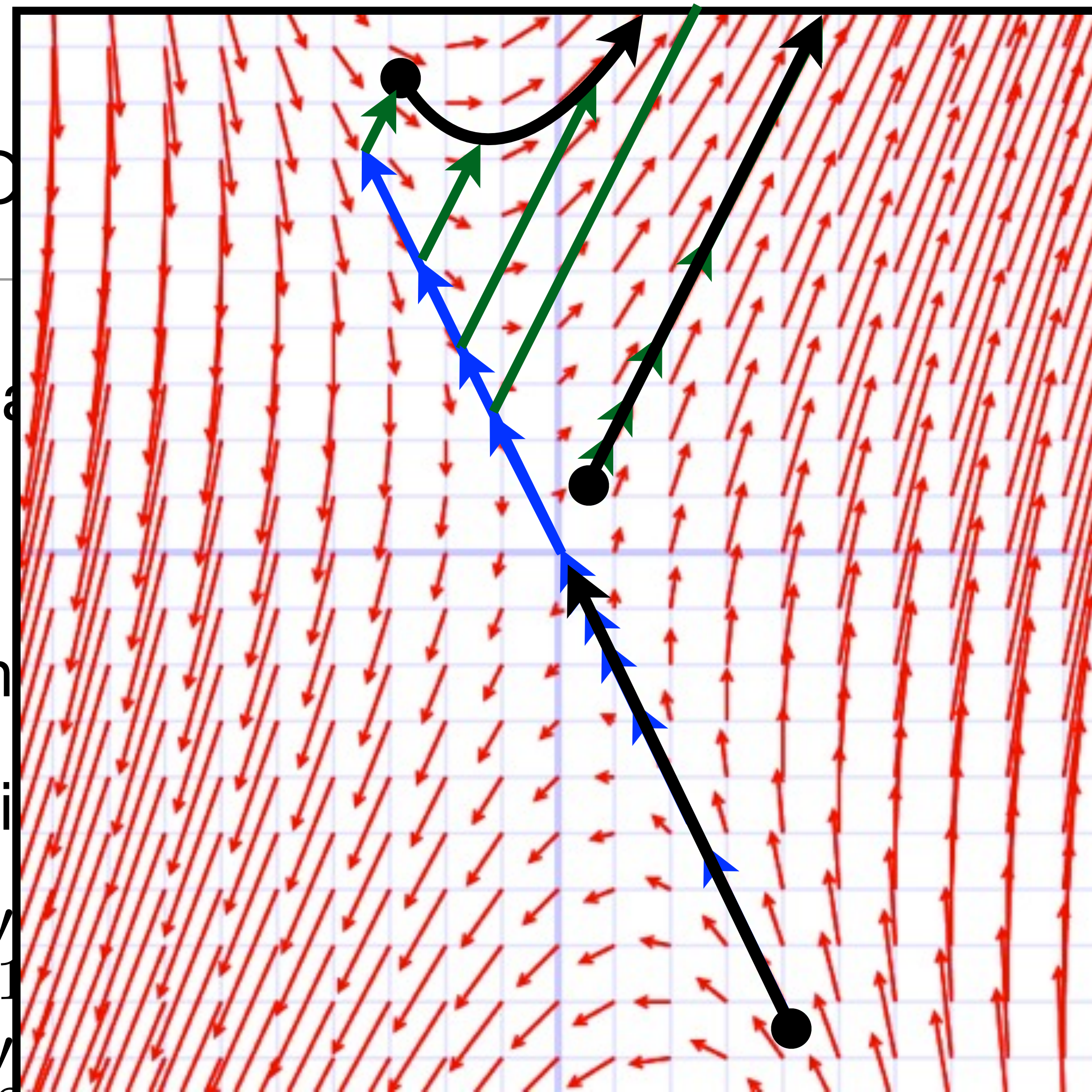

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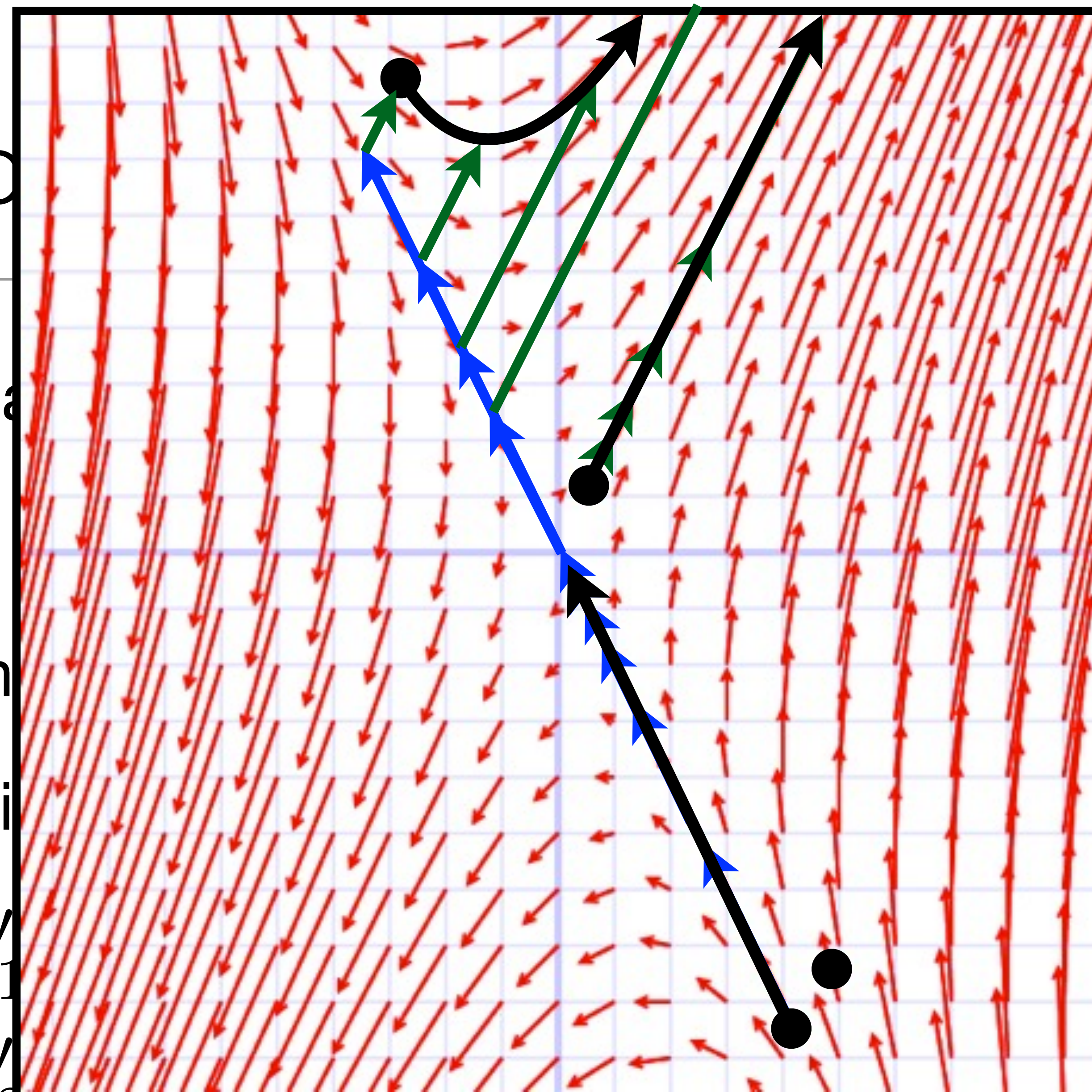

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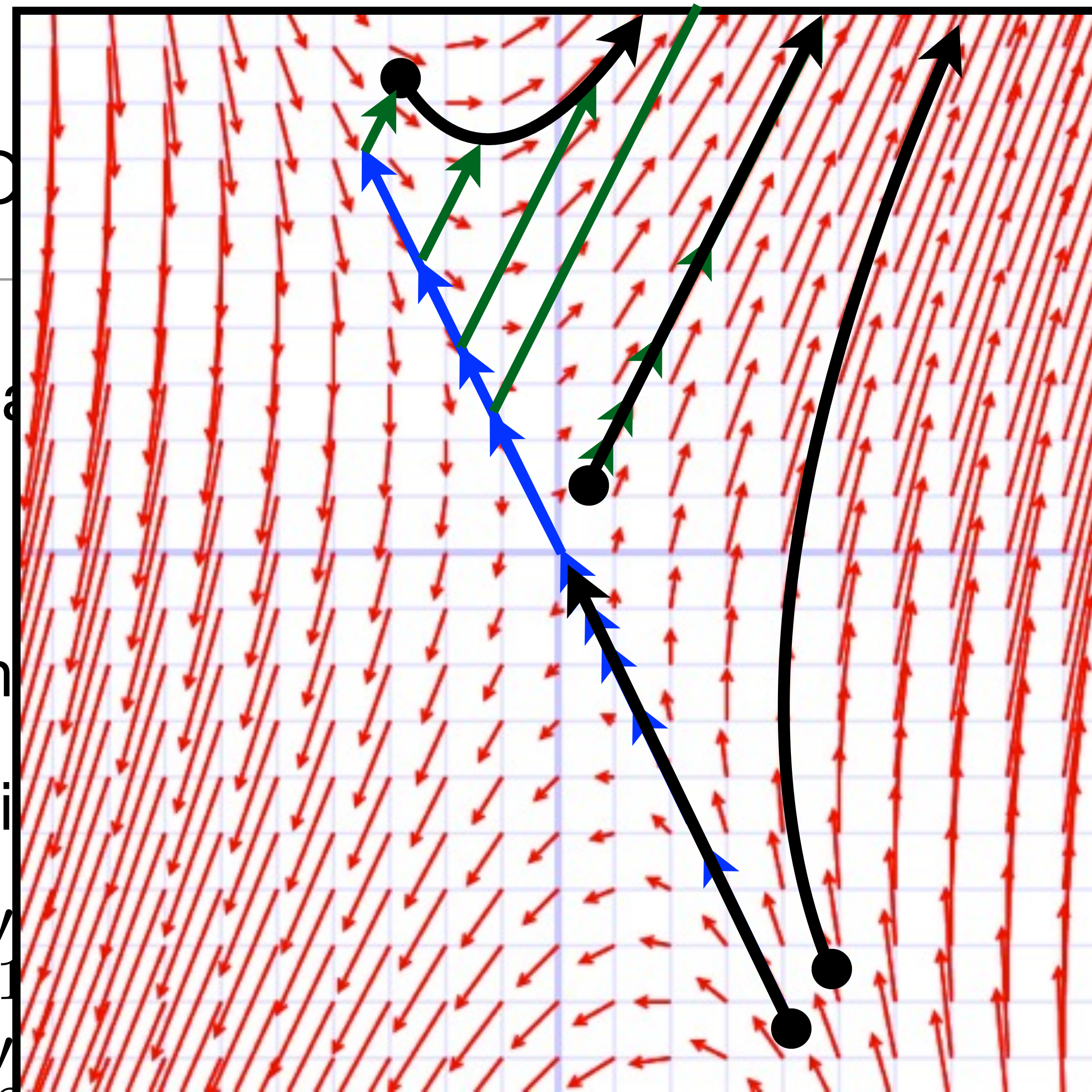

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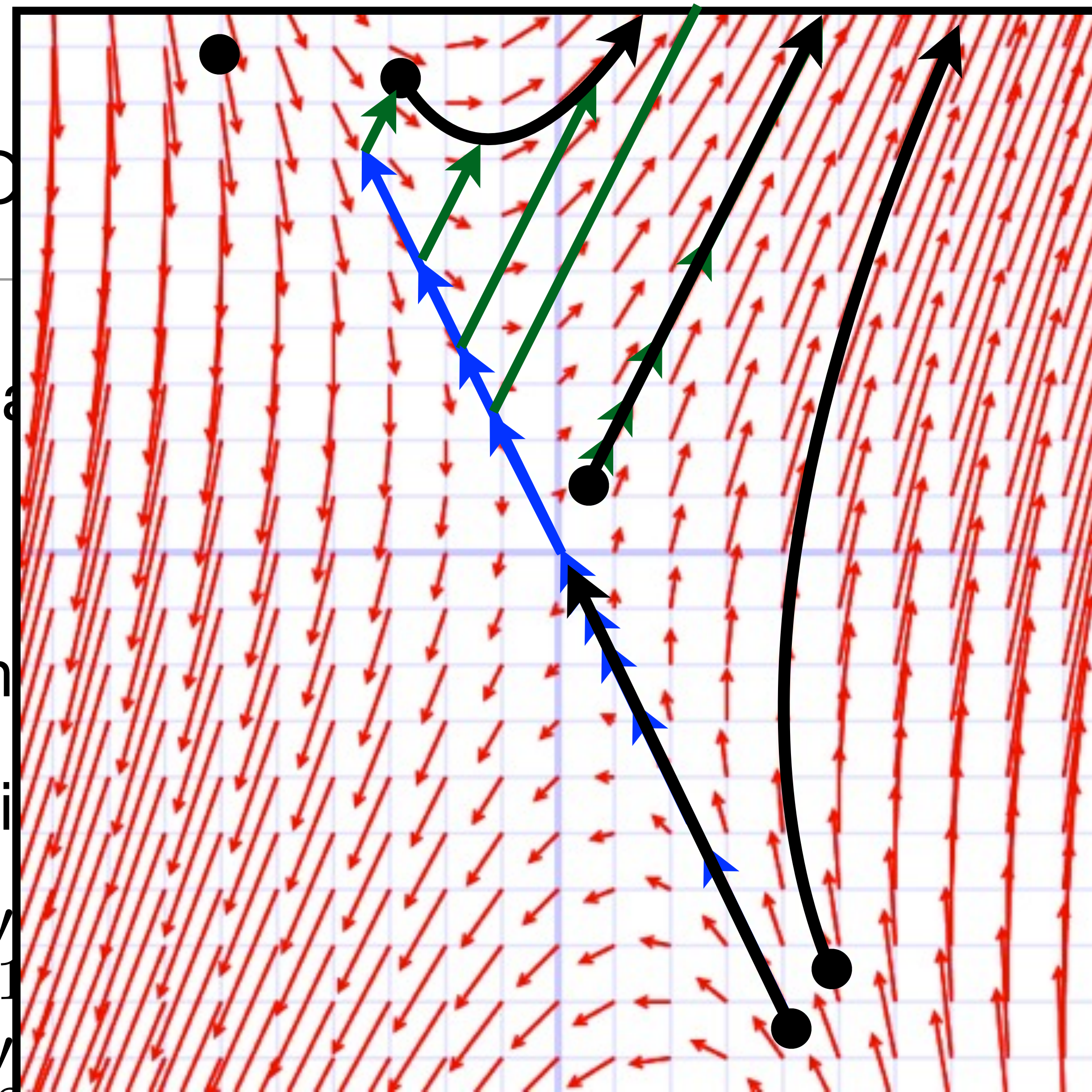

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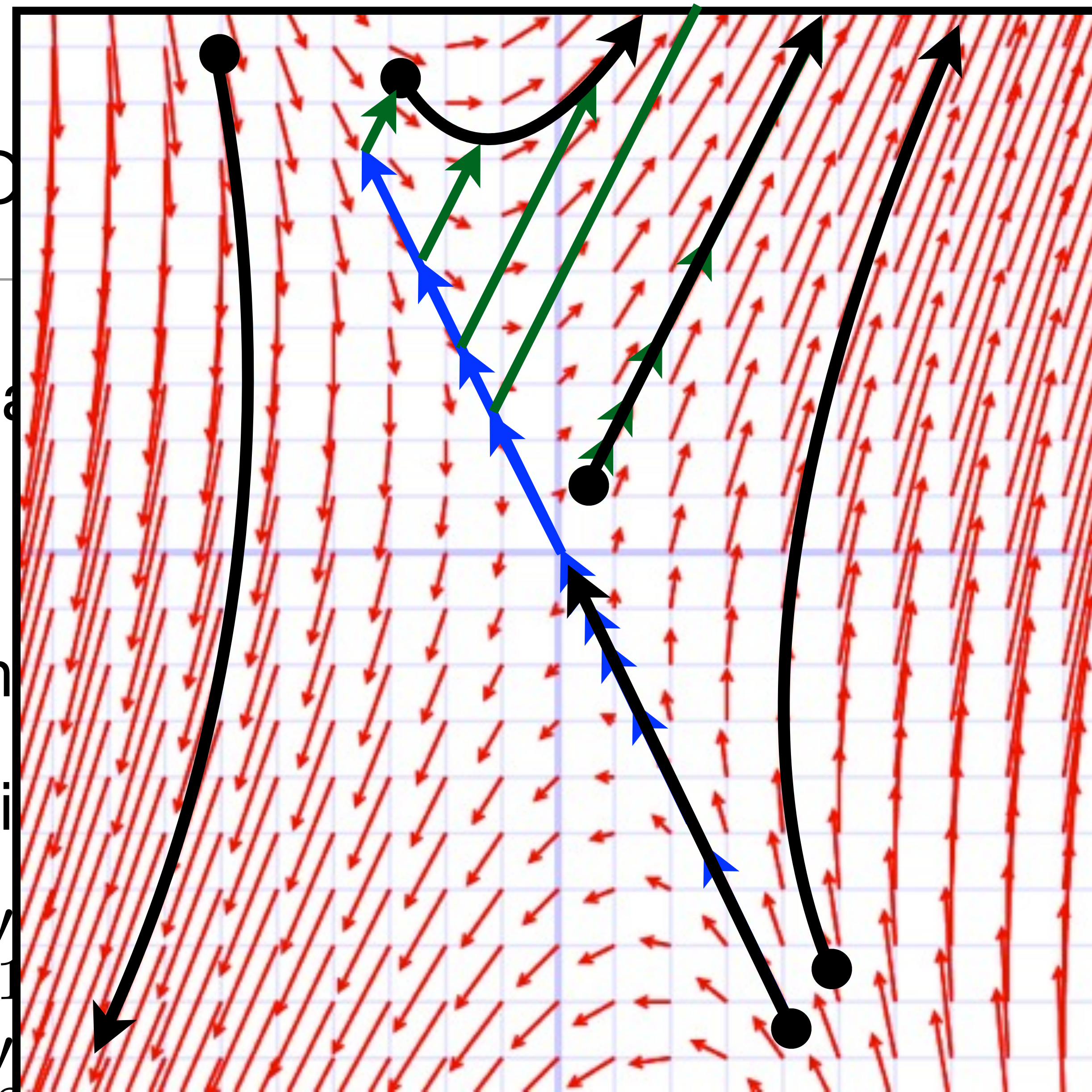

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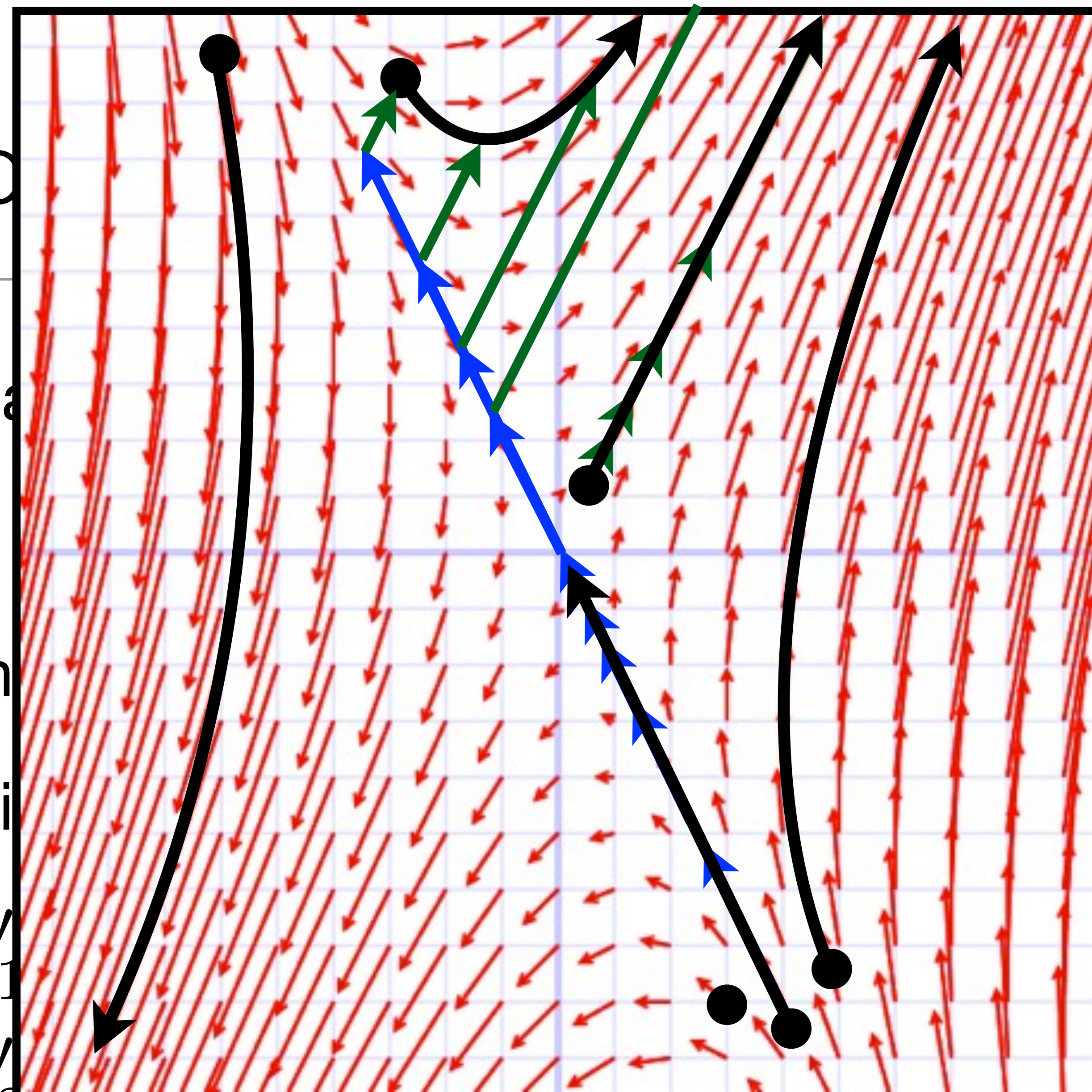

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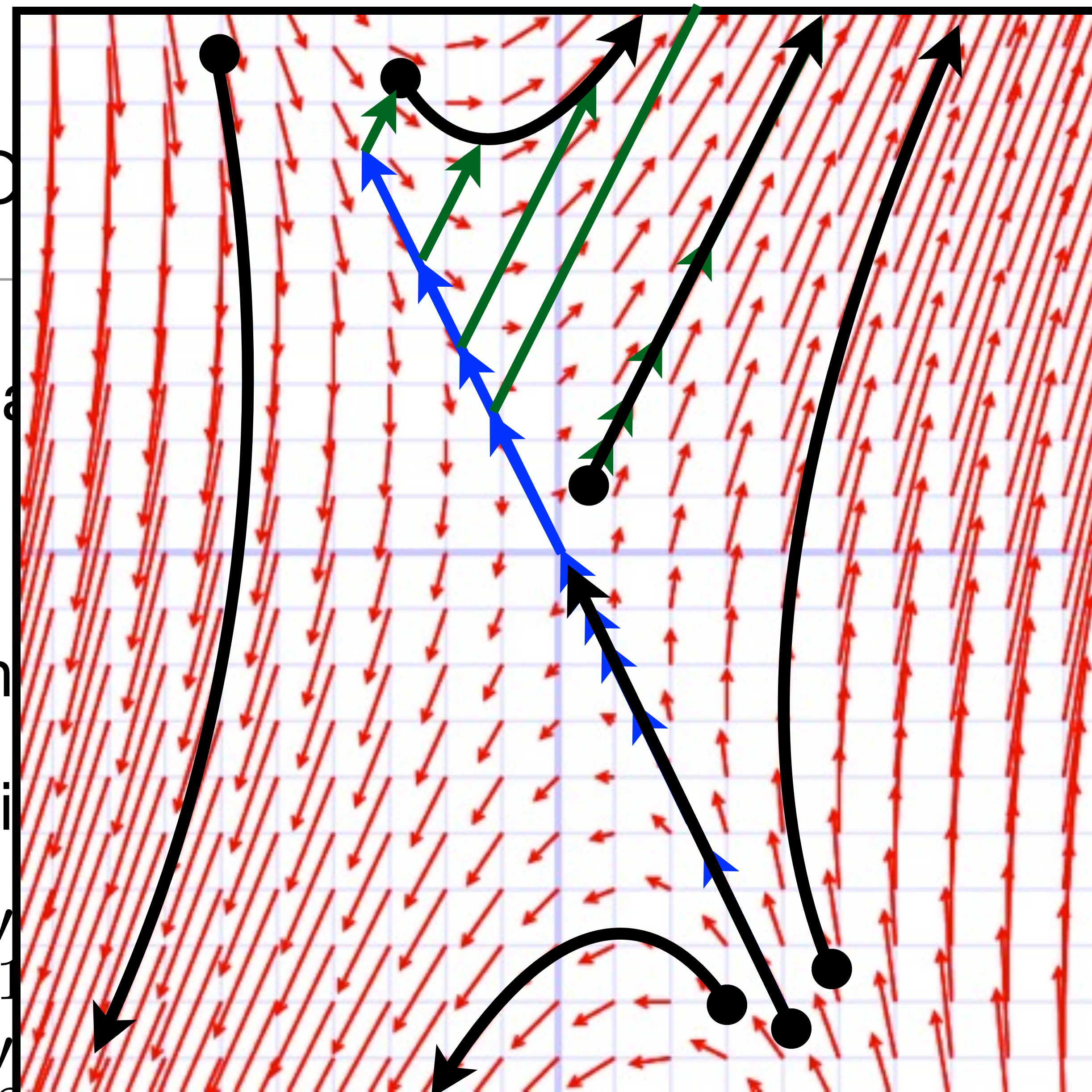

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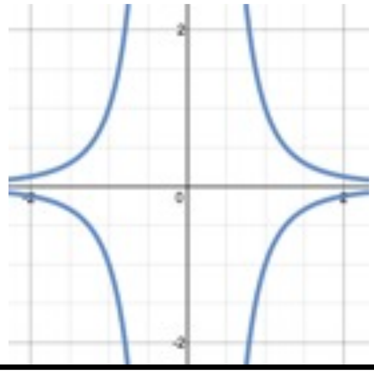

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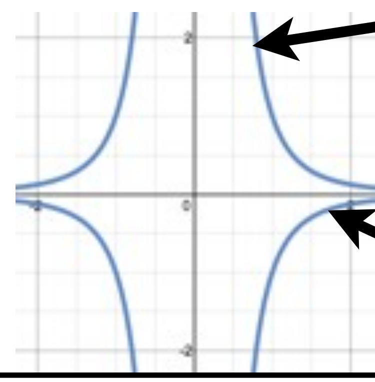

far from λ_2 axis

close to λ_1 axis

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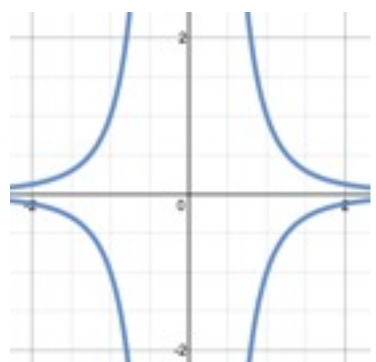

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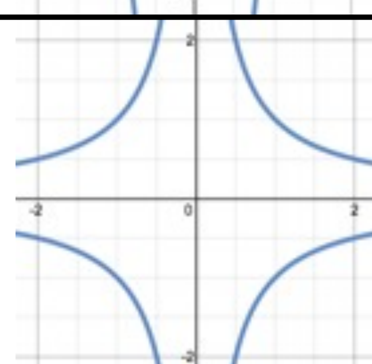

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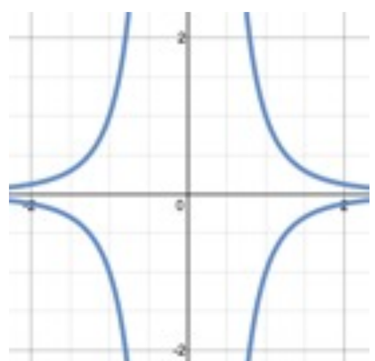

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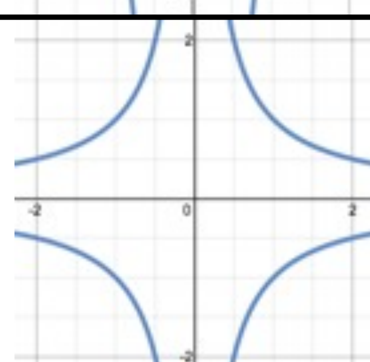

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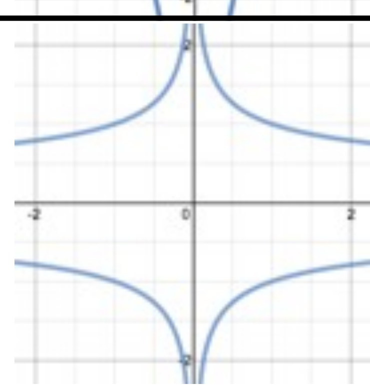

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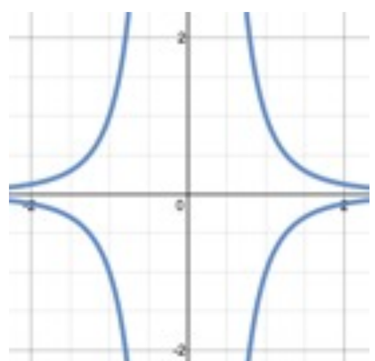

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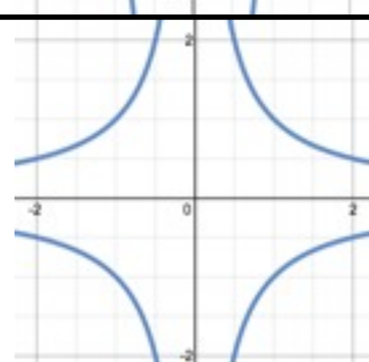

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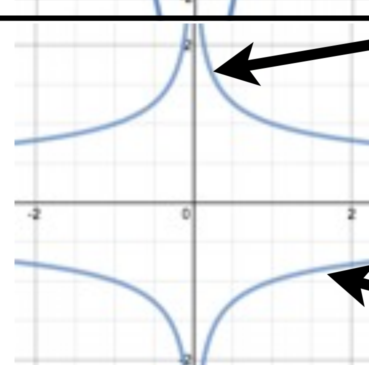

close to λ₂ axis

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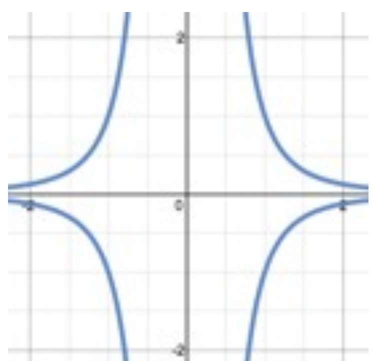

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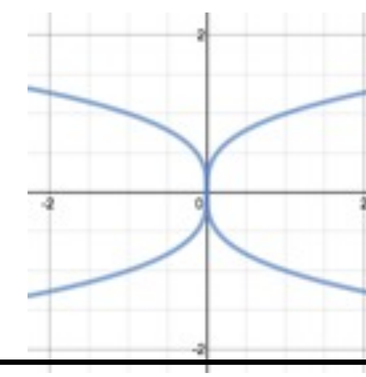

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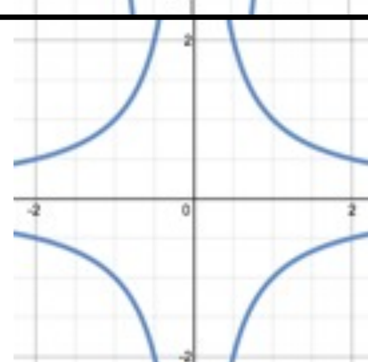

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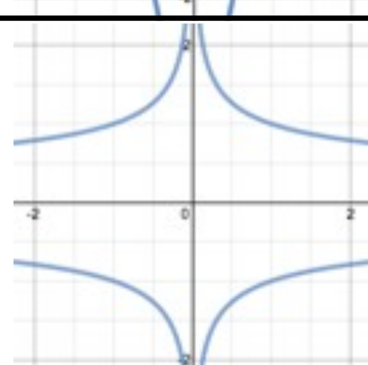

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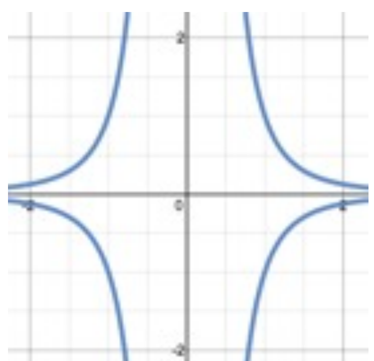

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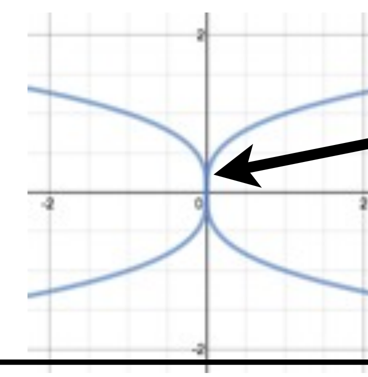

stays near λ_2 axis

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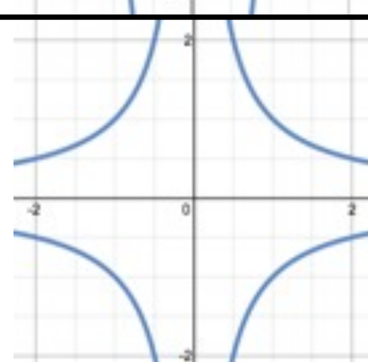

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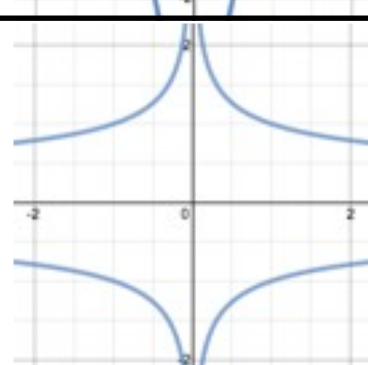

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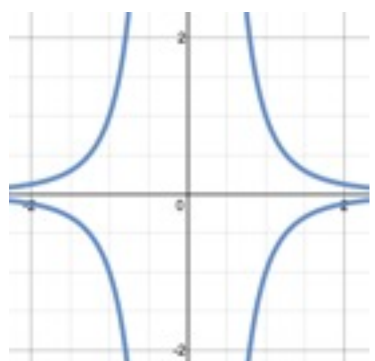

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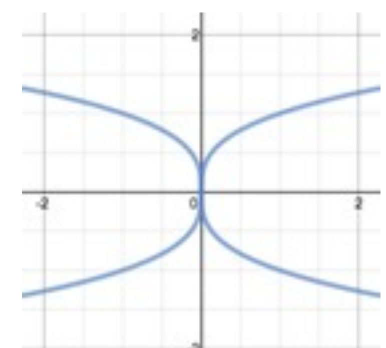

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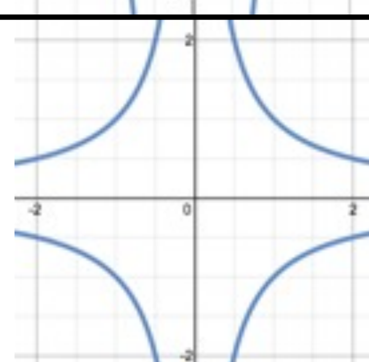

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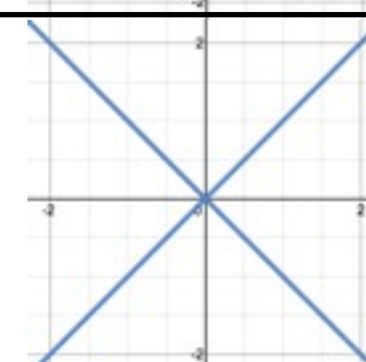

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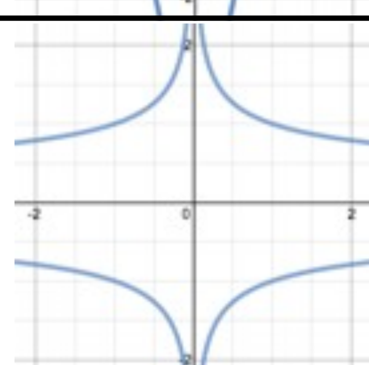

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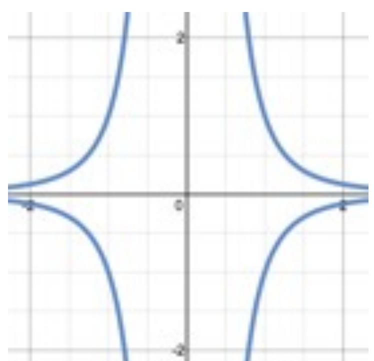

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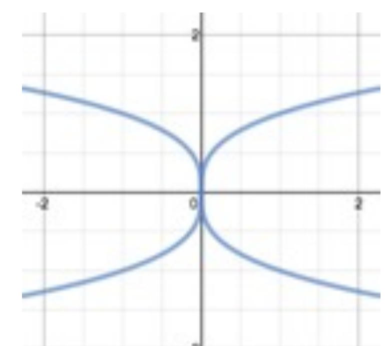

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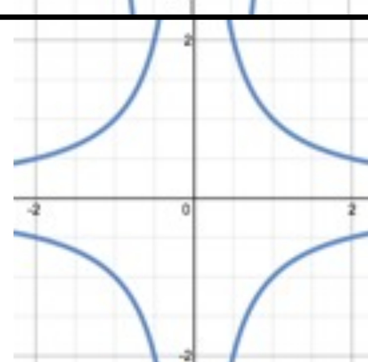

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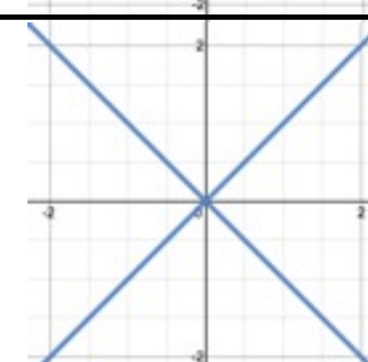

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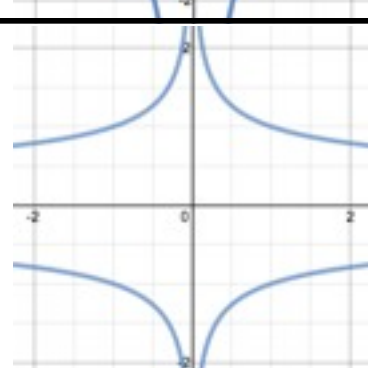

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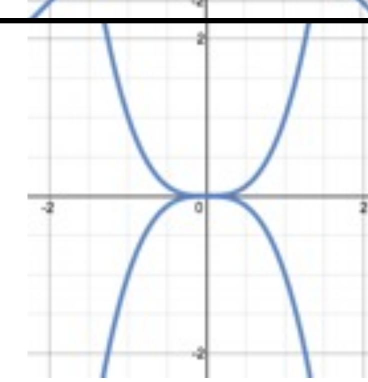

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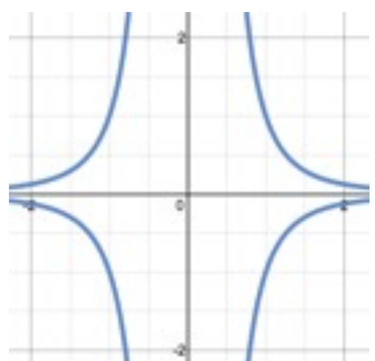

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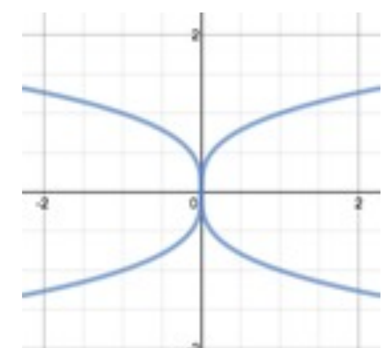

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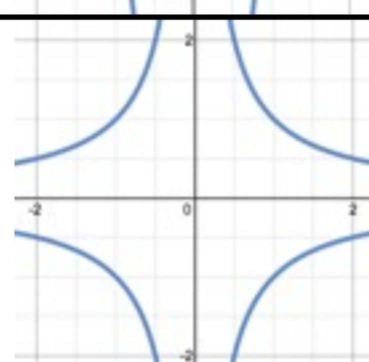

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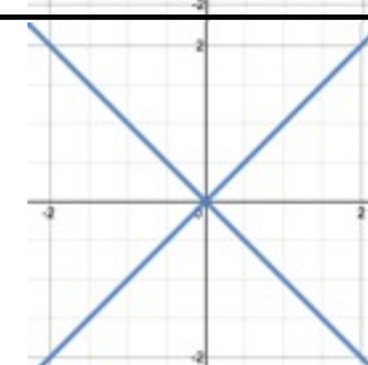

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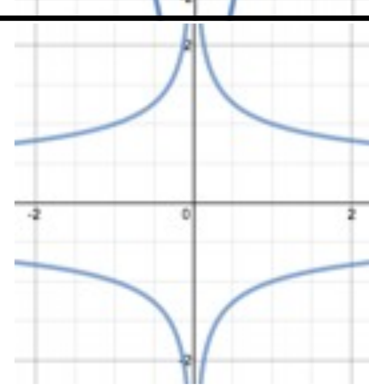

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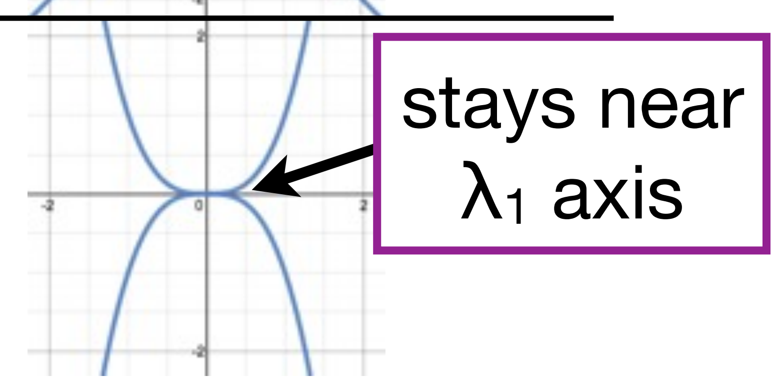

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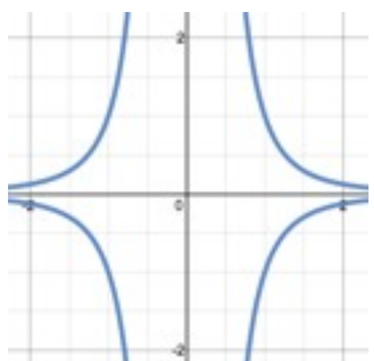

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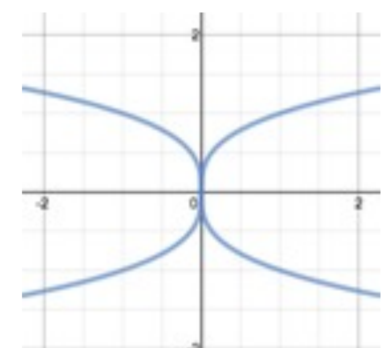

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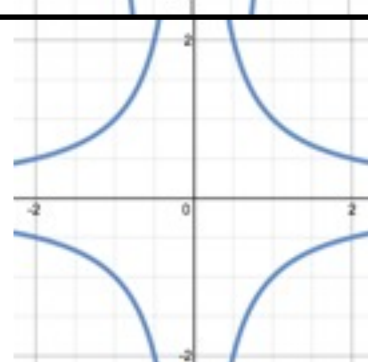

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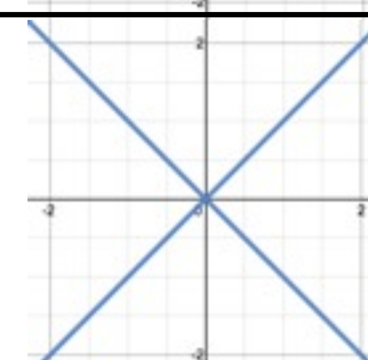

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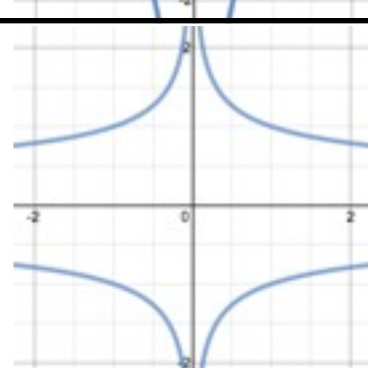

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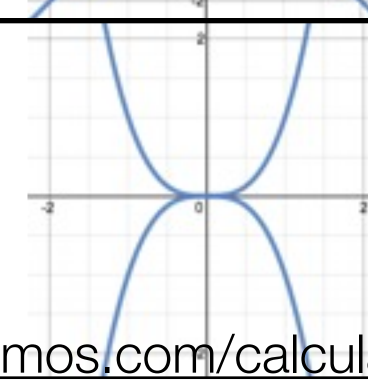

https://www.desmos.com/calculator/c4rhrgotmo

Shapes of solution curves in the phase plane

- With more complicated solutions (evectors off-axis), tilt shapes accordingly.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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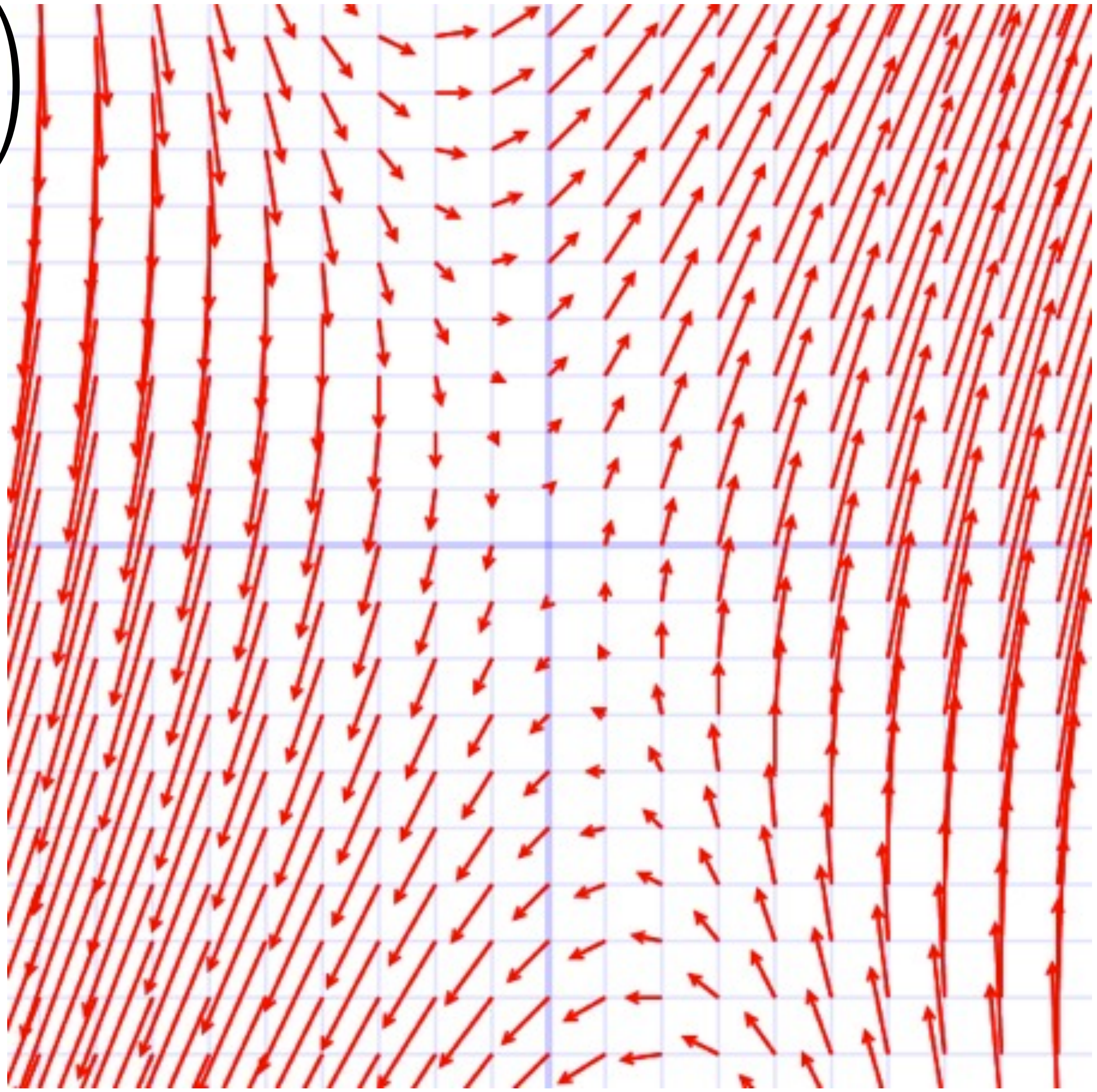

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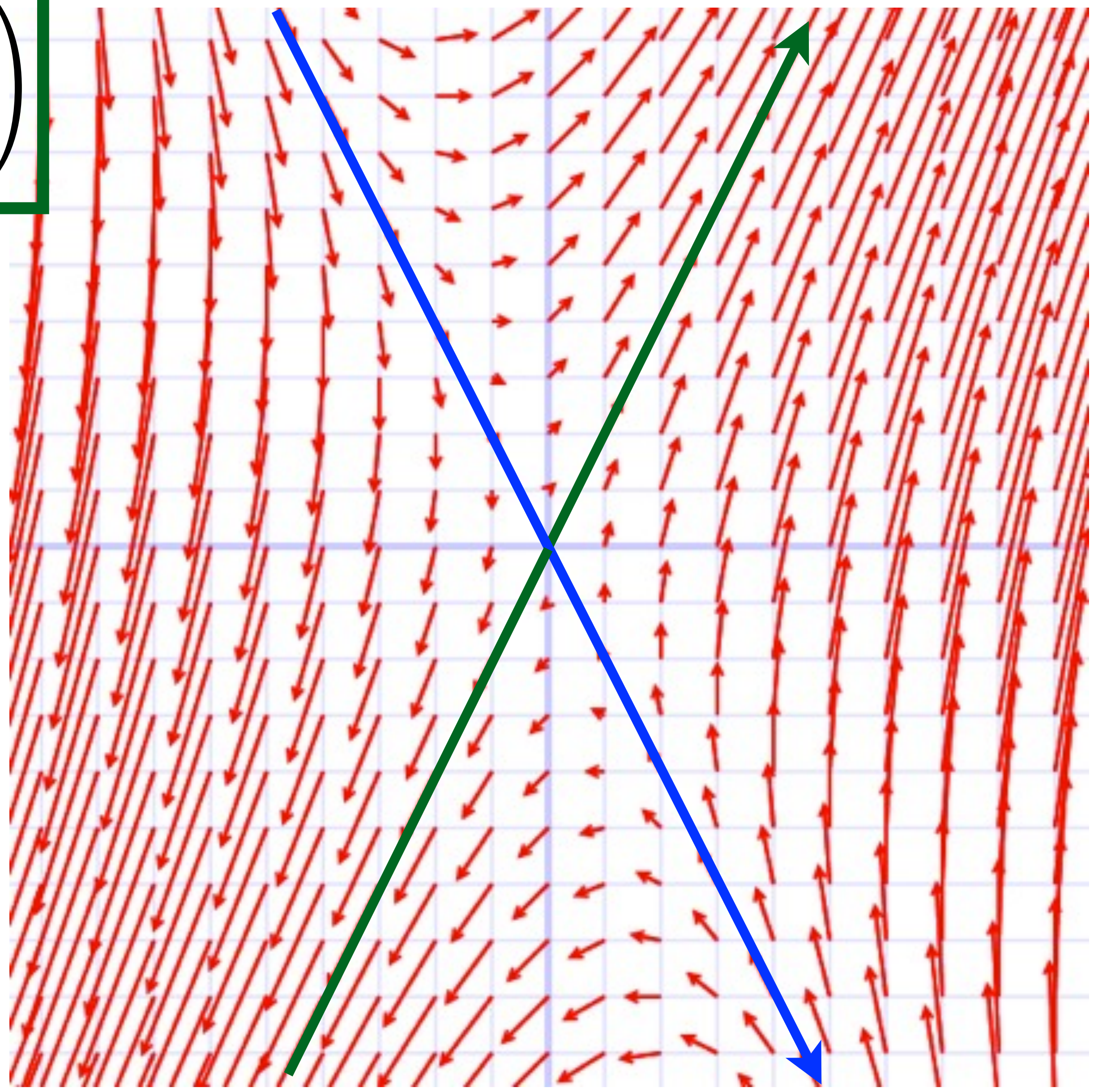

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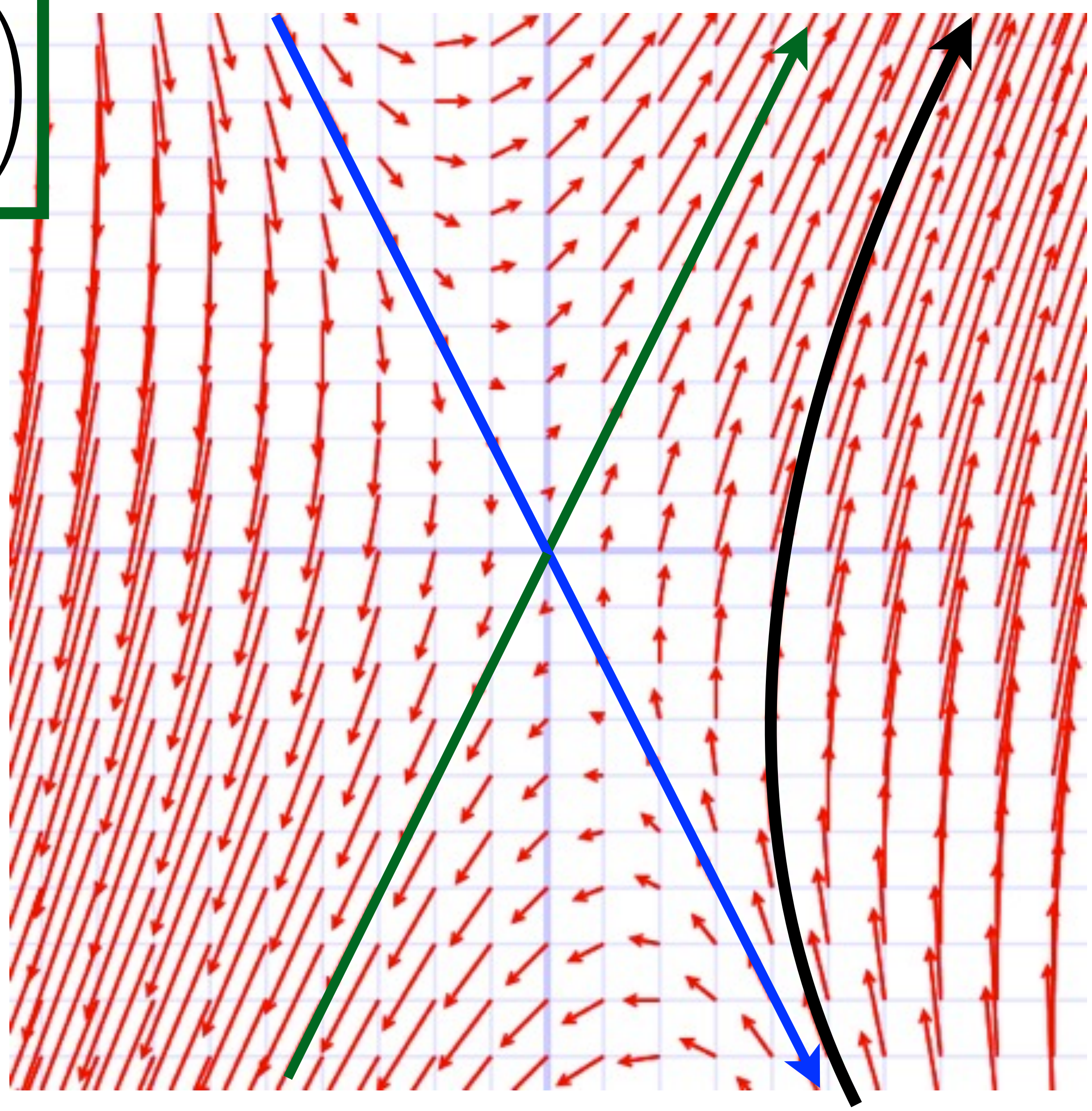

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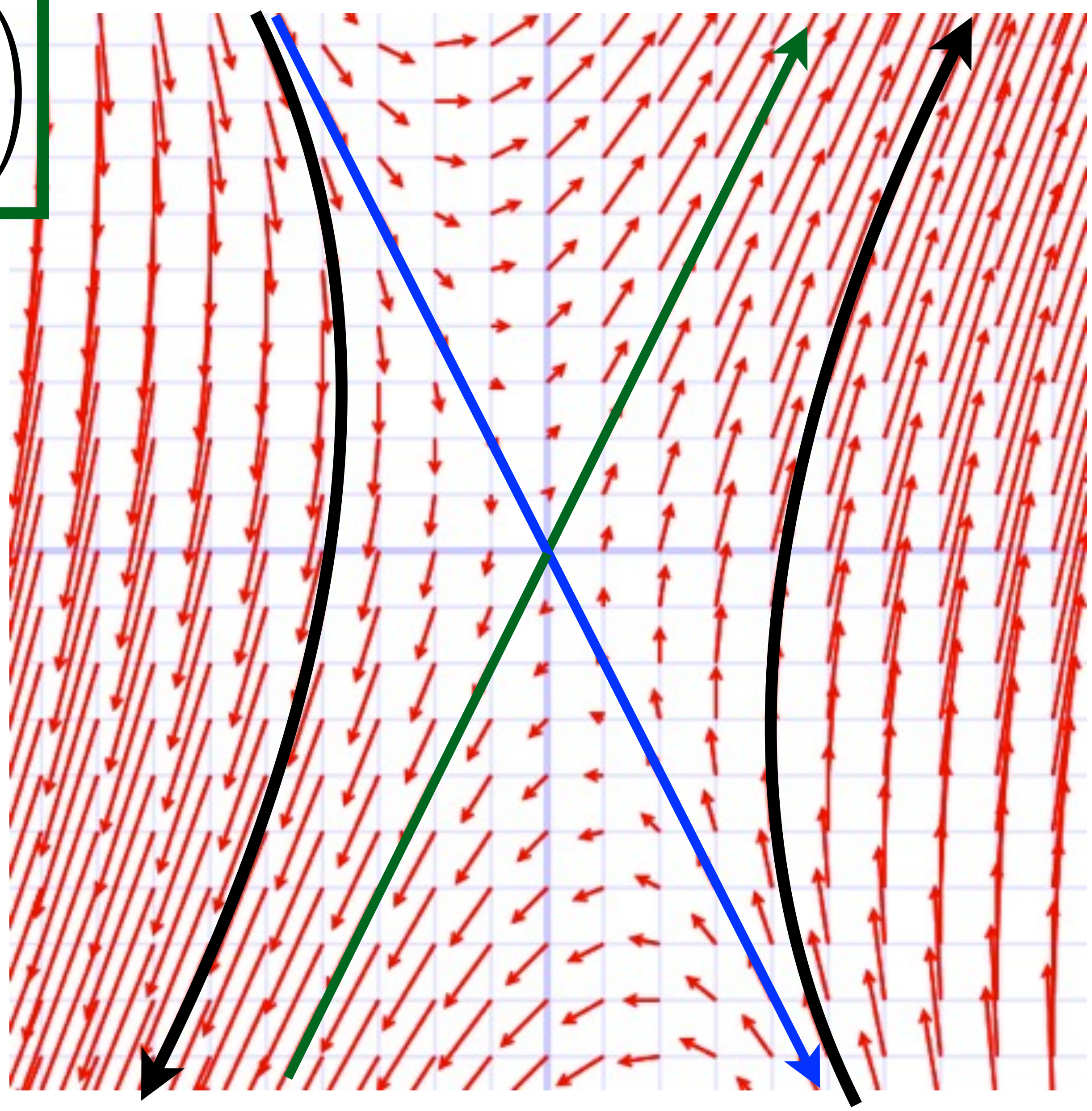

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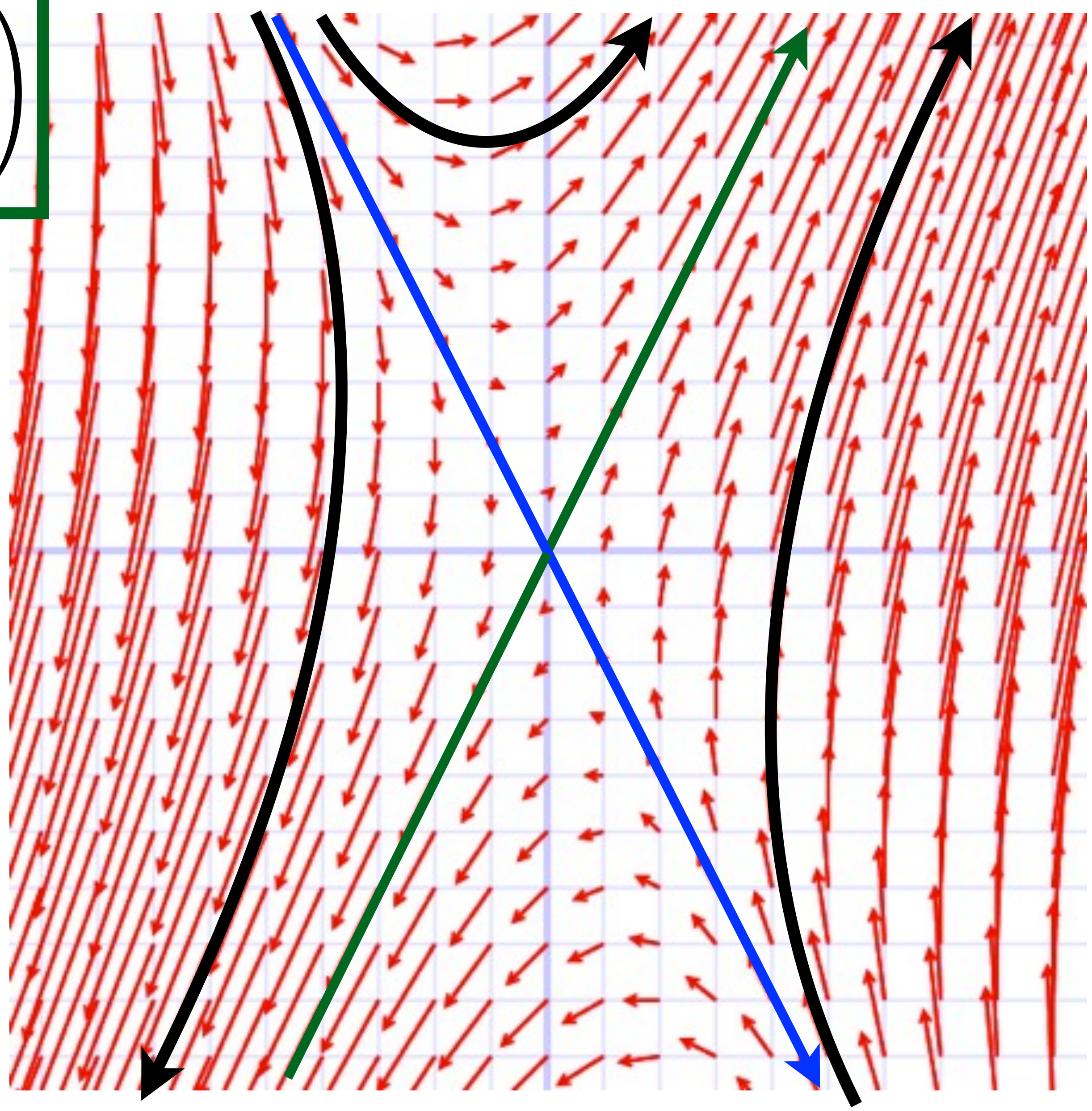

Shapes of solution curves in the phase plane

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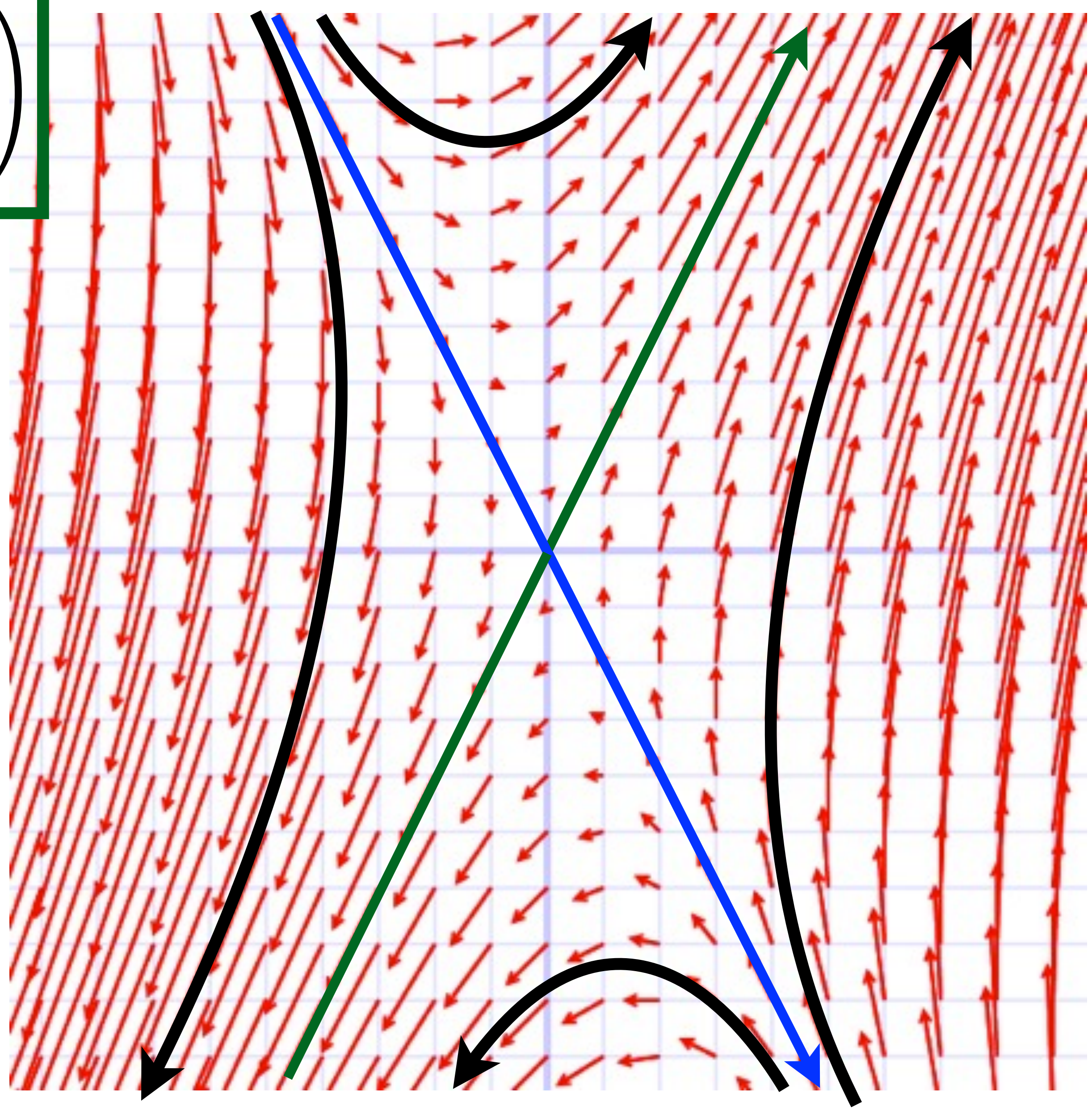

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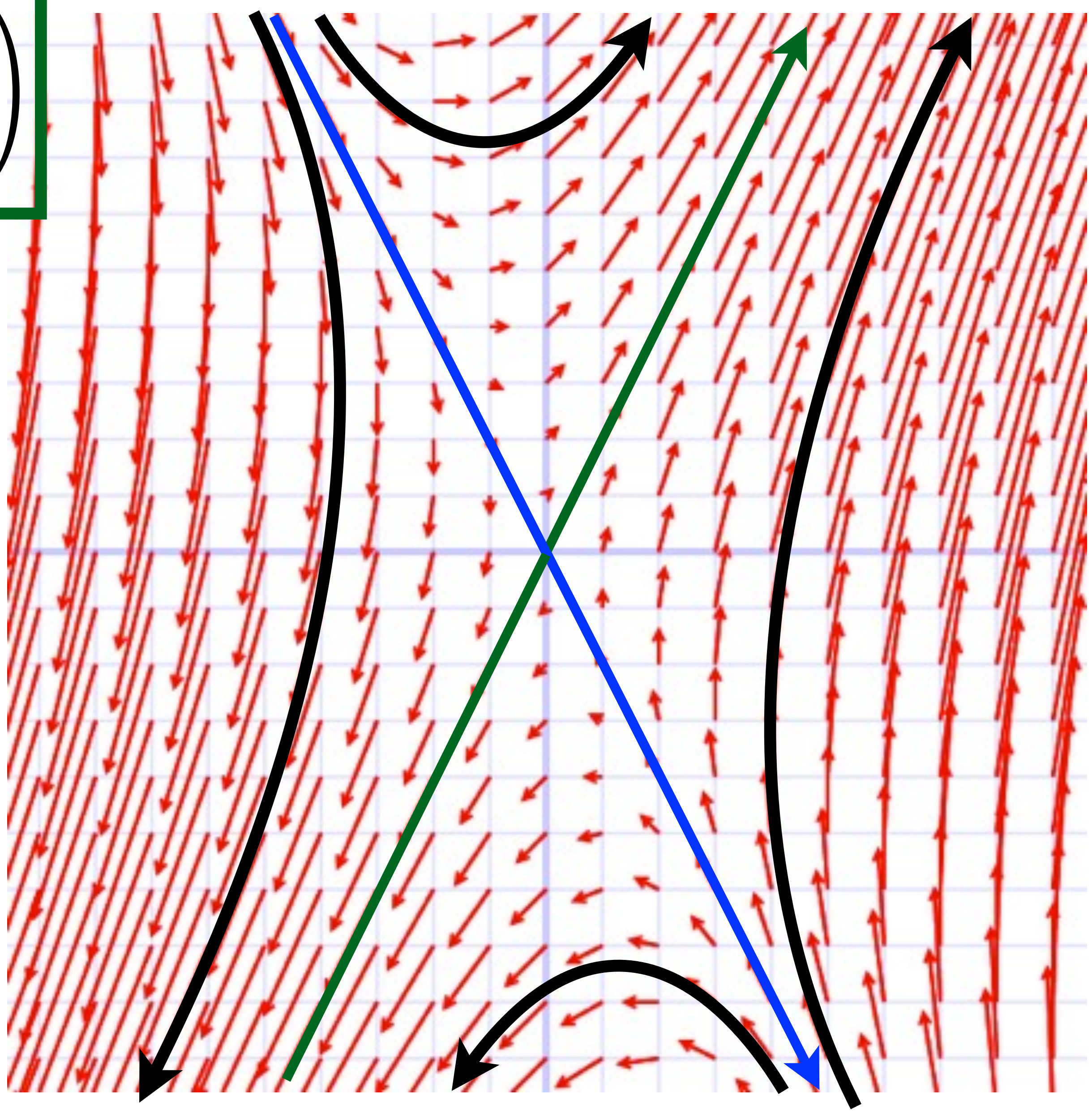


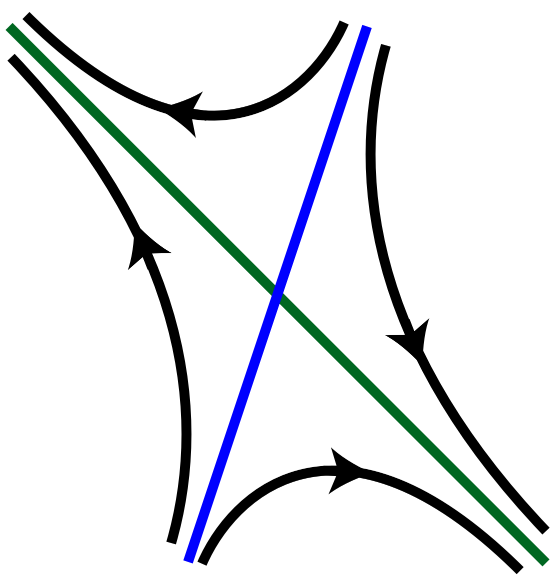
- Going forward in time, the blue component shrinks slower than the green component grows so solutions appear closer to blue "axis" than to green "axis"

Shapes of solution curves in the phase plane

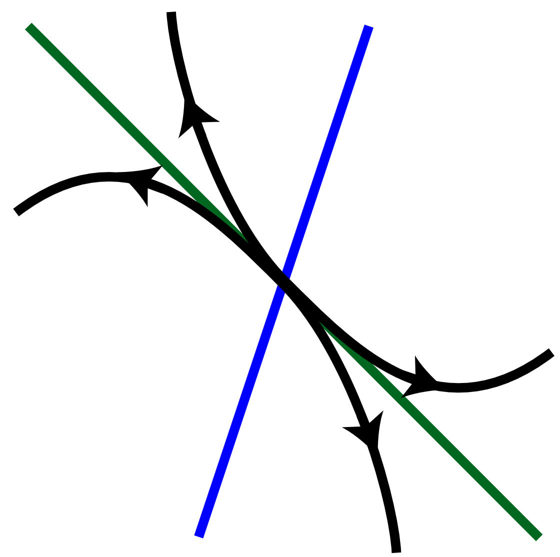
- Which phase plane matches the general solution

$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \; ?$$

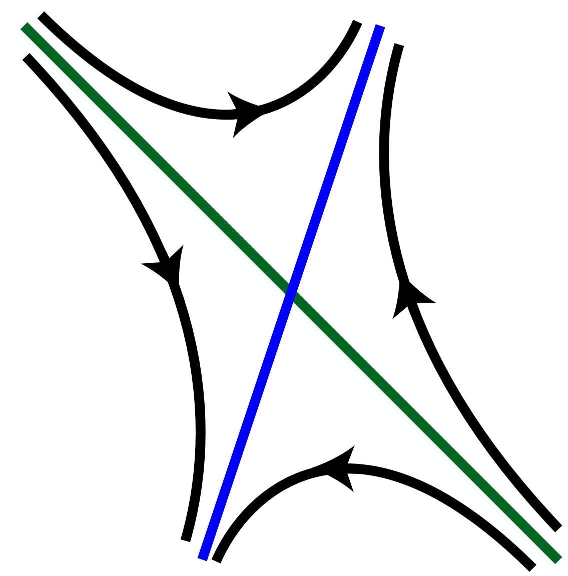
(A)



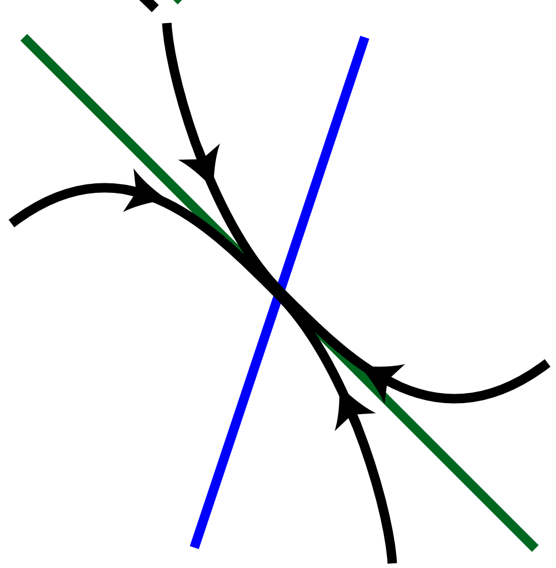
(B)



(C)



(D)



(E) Explain, please.

Shapes of solution curves in the phase plane

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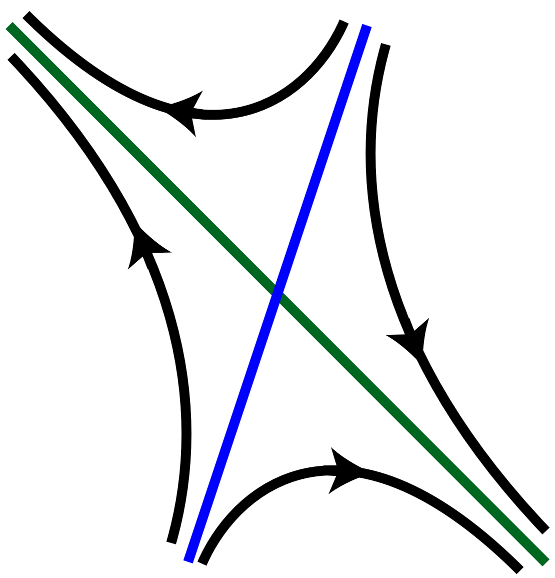

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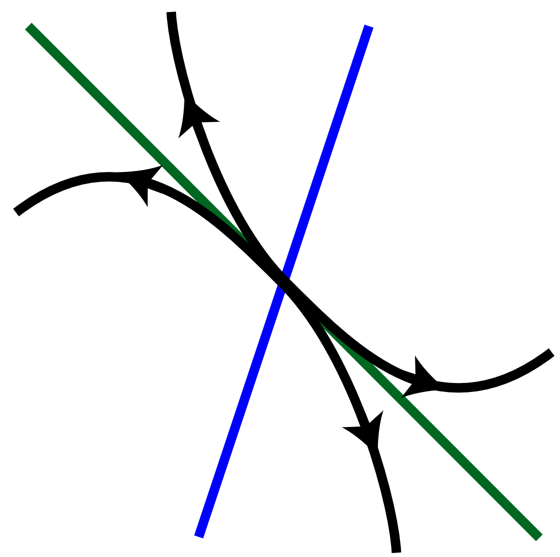

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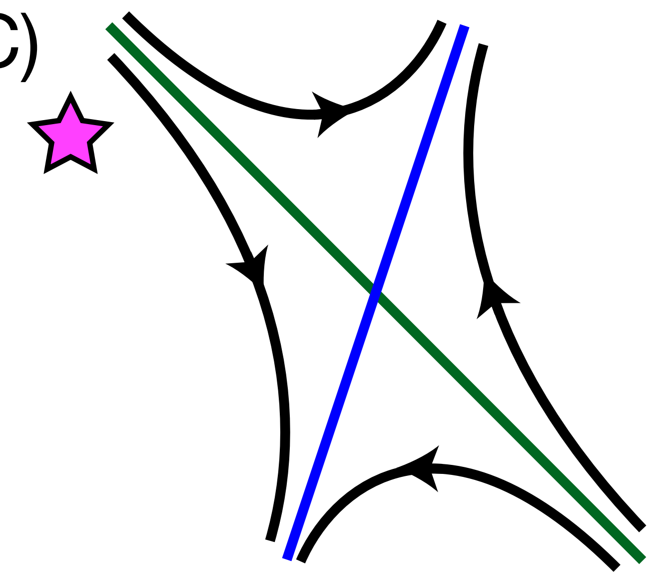

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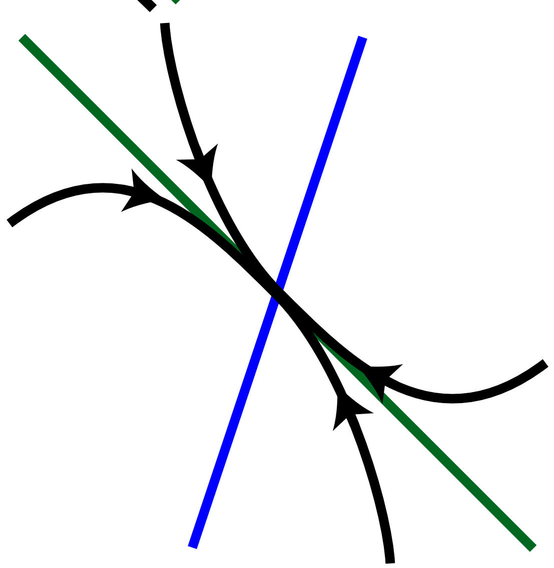

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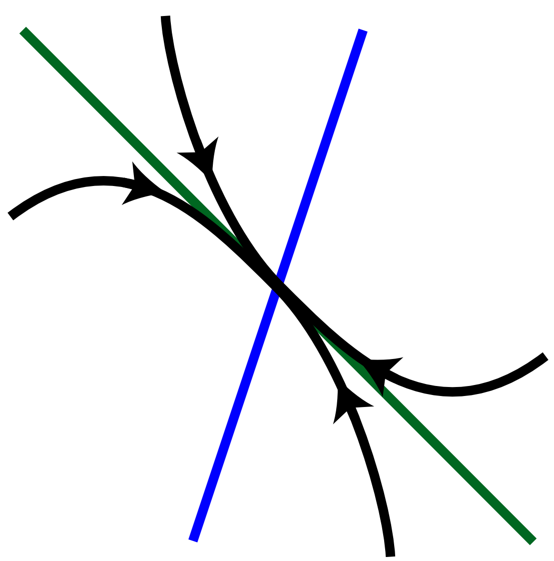

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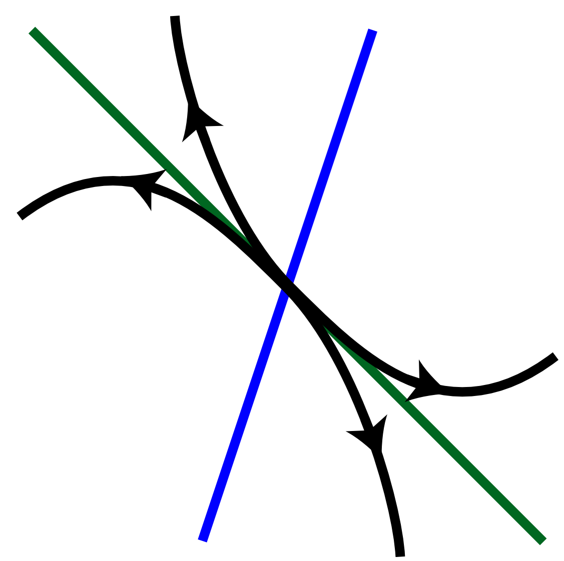

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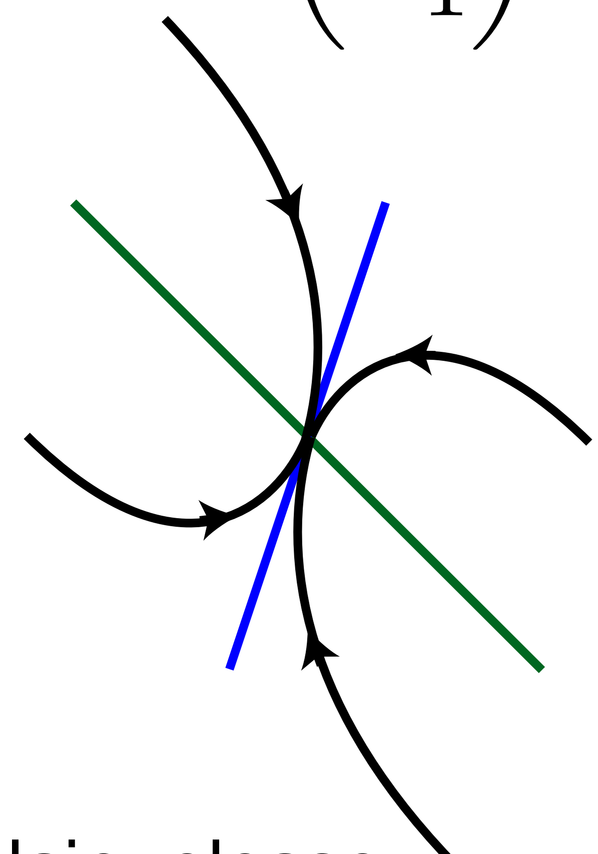

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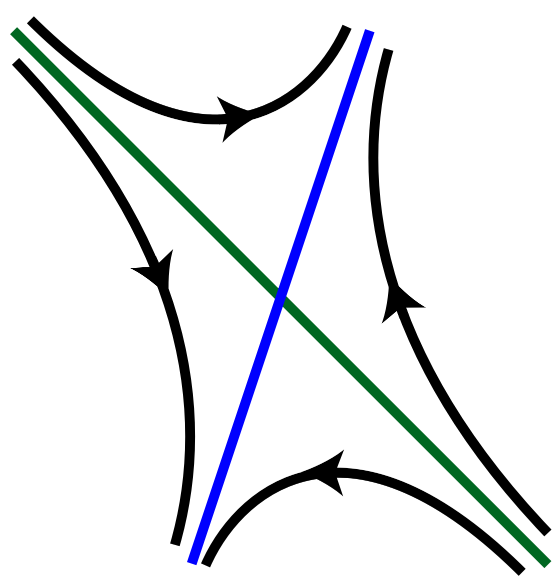

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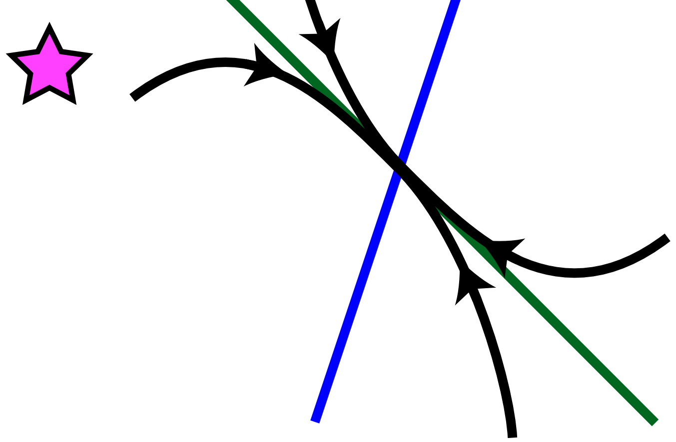
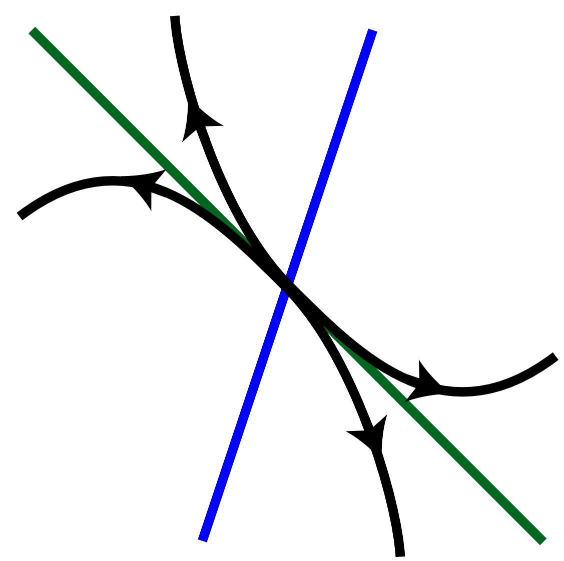
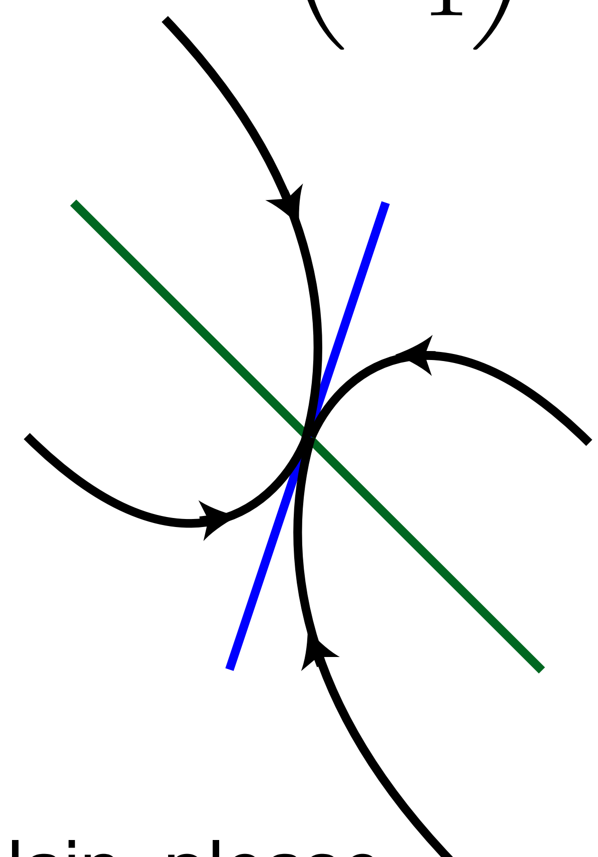


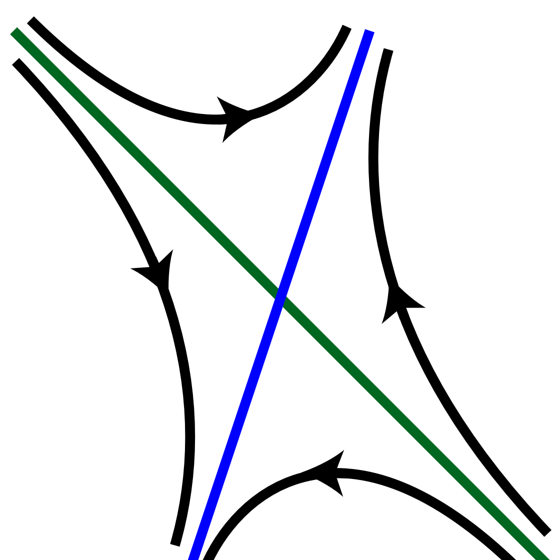
(B)



(C)



(D)



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Complex eigenvalues (7.6) - example

- Find the general solution to $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x}$.
  - The eigenvalues are

    (A) $\lambda = 1 \pm 2i$

    (B) $\lambda = -1, 3$

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$$A - \lambda_1 I = \begin{pmatrix} 1-(1+2i) & 1 \\ -4 & 1-(1+2i) \end{pmatrix}$$

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- We could just write down a (complex valued) general solution:

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$$x_1'' - 2x_1' + 5x_1 = 0$$

Complex eigenvalues (7.6) - example

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$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) \right.$$

Complex eigenvalues (7.6) - example

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$$\mathbf{x}(\mathbf{t}) = C_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + C_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

- But we want real valued solutions.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \mathbf{x} \qquad \Rightarrow \qquad \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -4x_1 + x_2 \end{aligned}$$

$$x_1(t) = e^t (C_1 \cos(2t) + C_2 \sin(2t))$$

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Complex eigenvalues (7.6) - general case

- Find e-values, $\lambda = \alpha \pm \beta i$, and e-vectors, $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

- Write down solution:

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Complex eigenvalues (7.6) - example

- Suppose you find eigenvalue $\lambda = 2\pi i$ and eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and, for some initial value problem,

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- Note: the initial condition was carefully chosen so that C_1=1 and C_2=0.

Complex eigenvalues (7.6) - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

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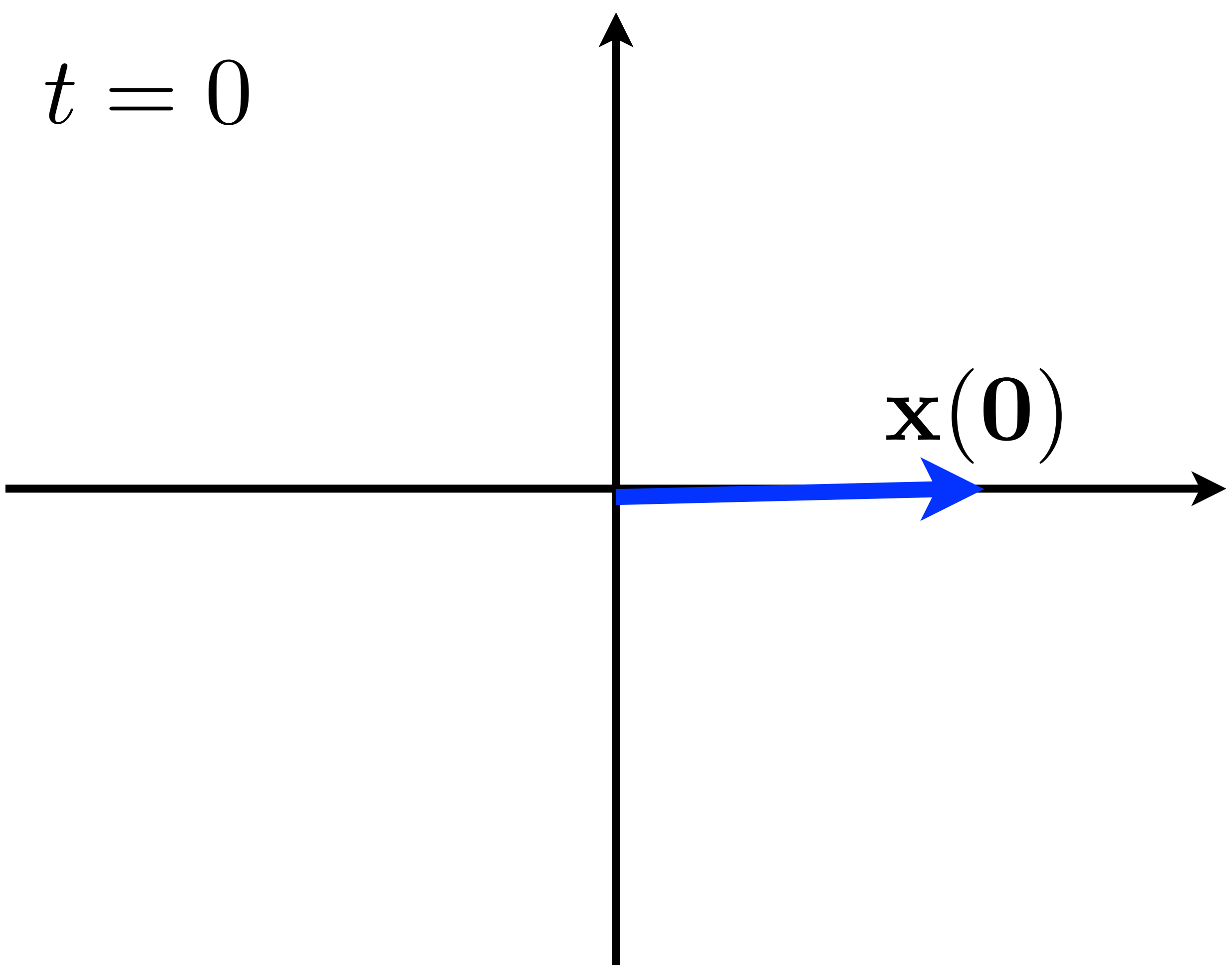

- What happens as t increases?

Complex eigenvalues (7.6) - example

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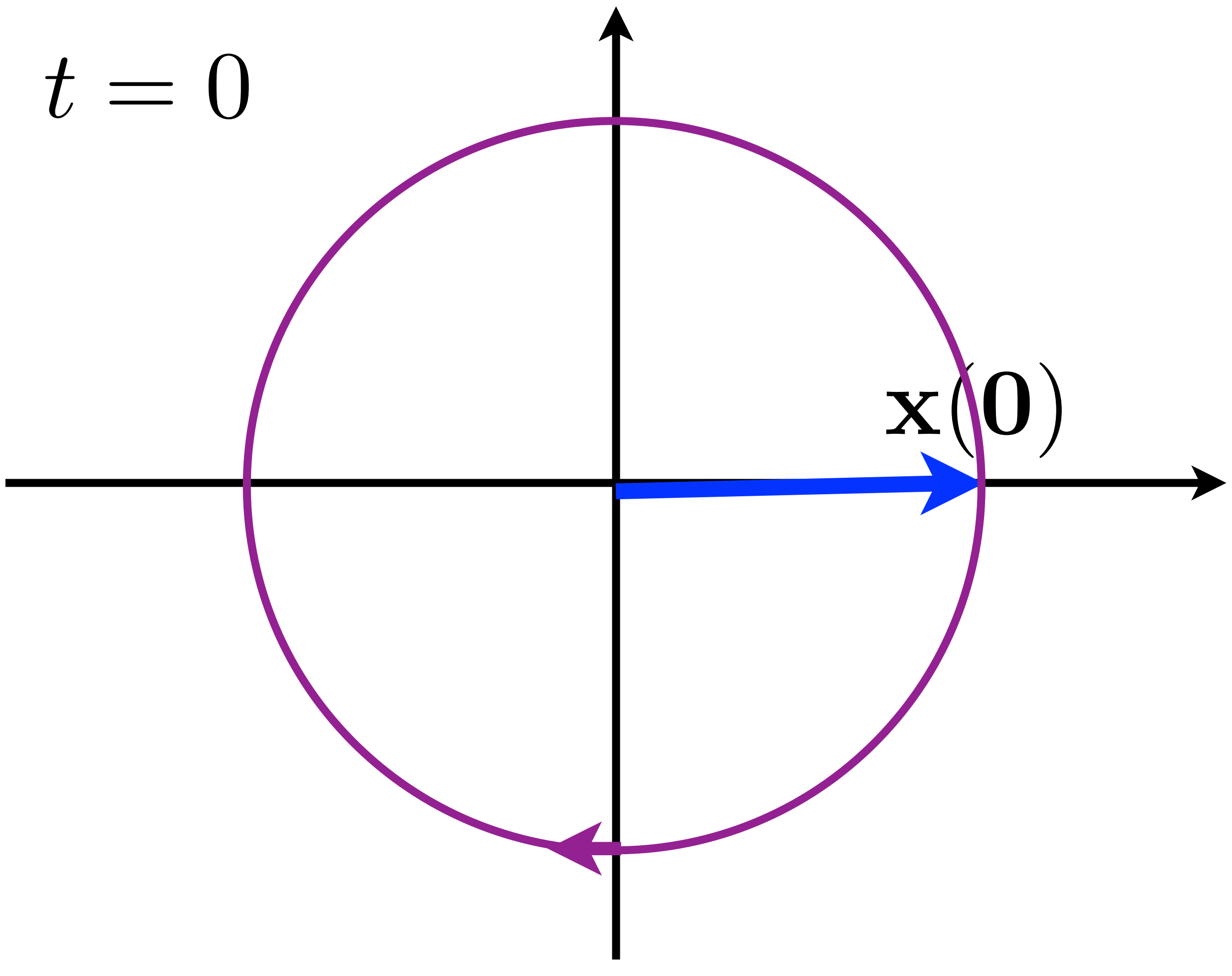

- What happens as t increases?

  (A) The vector rotates clockwise.

Complex eigenvalues (7.6) - example

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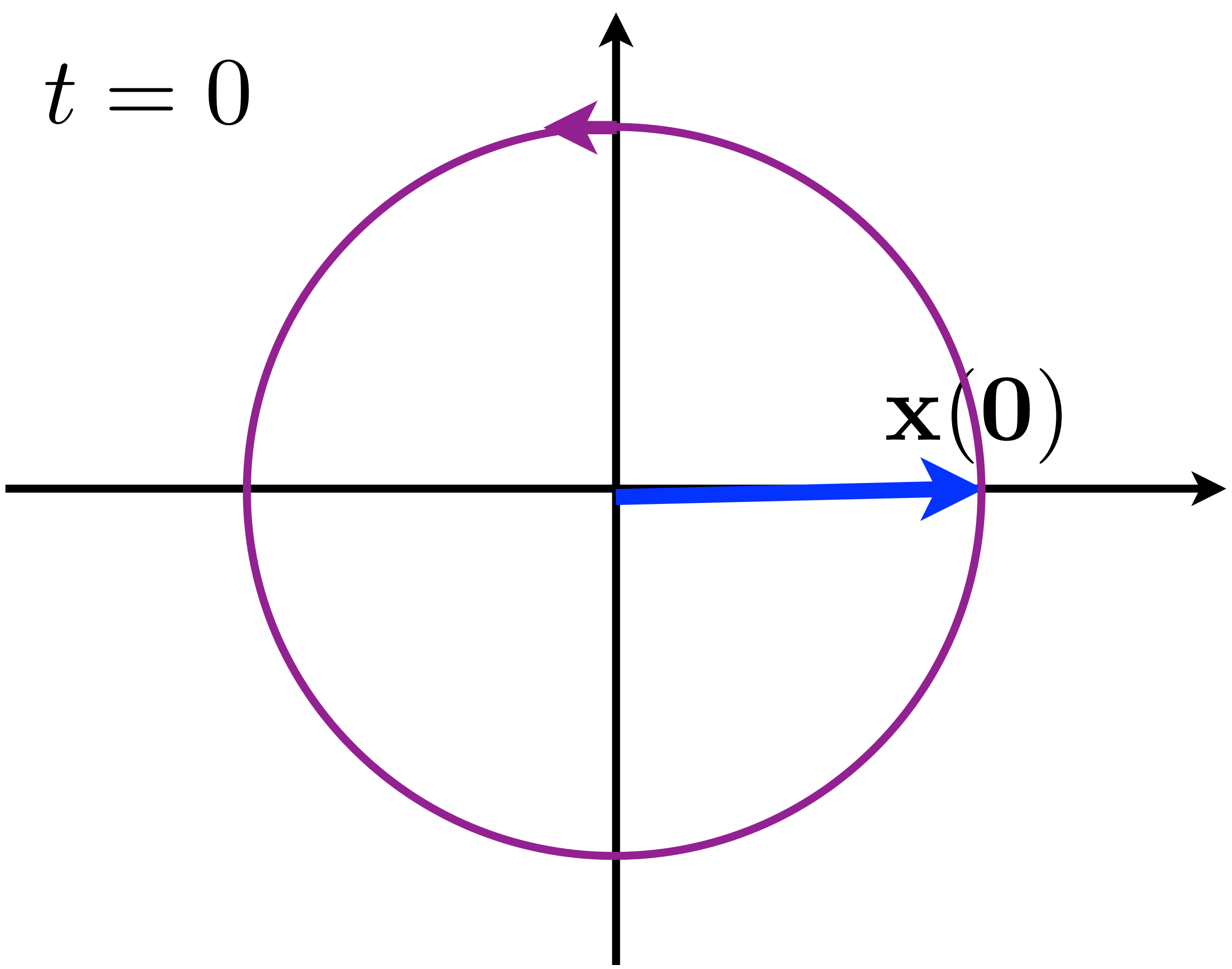

- What happens as t increases?
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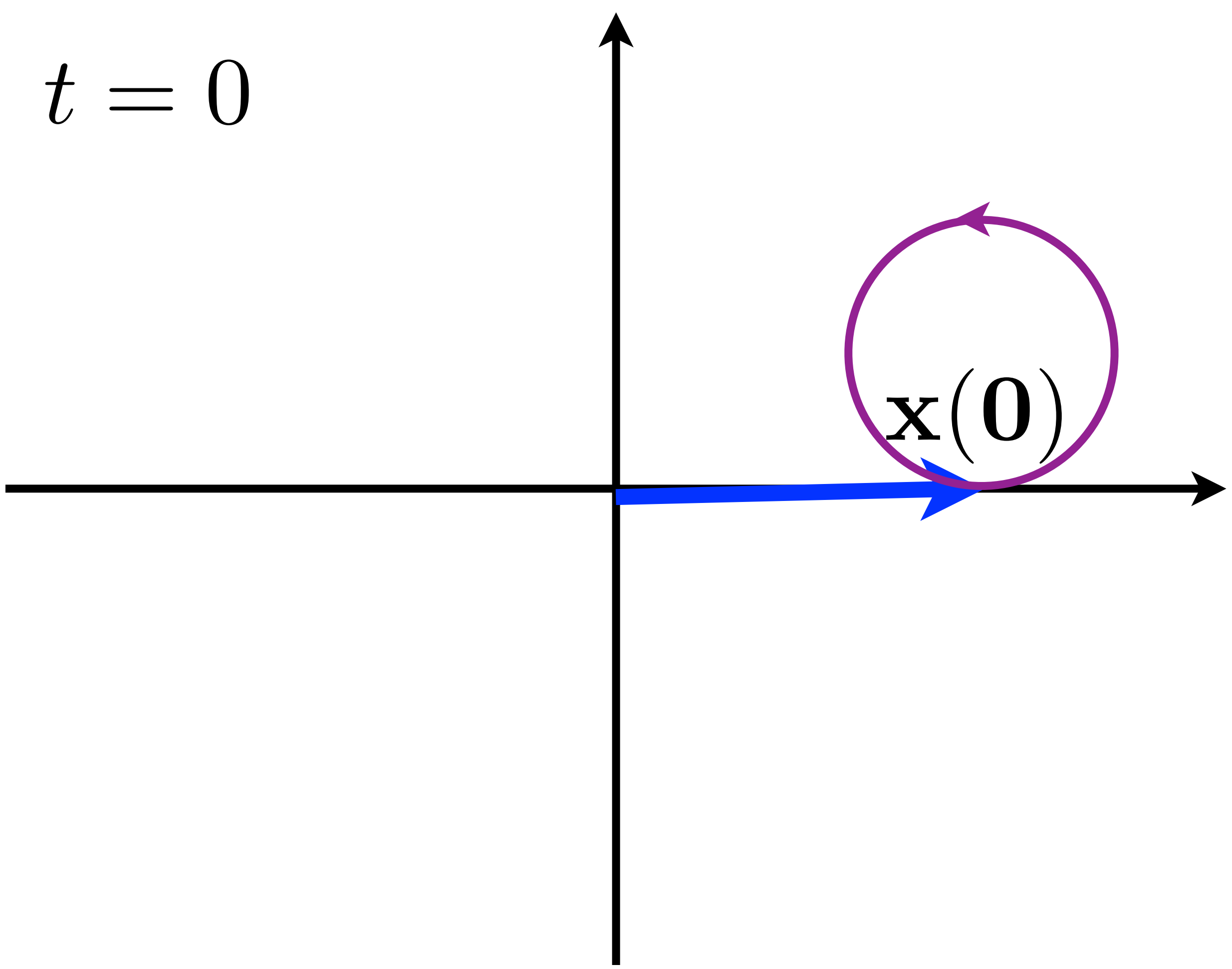

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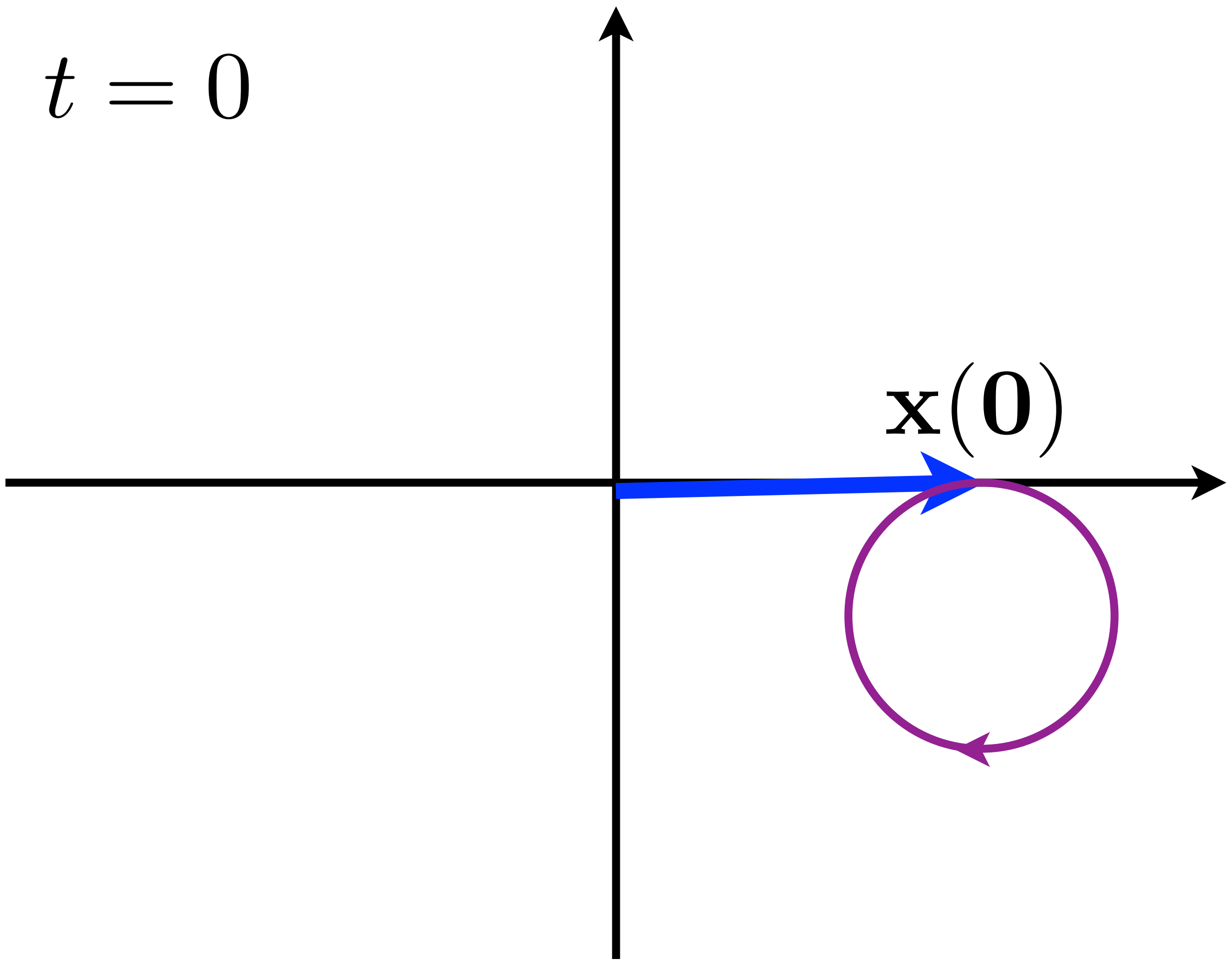

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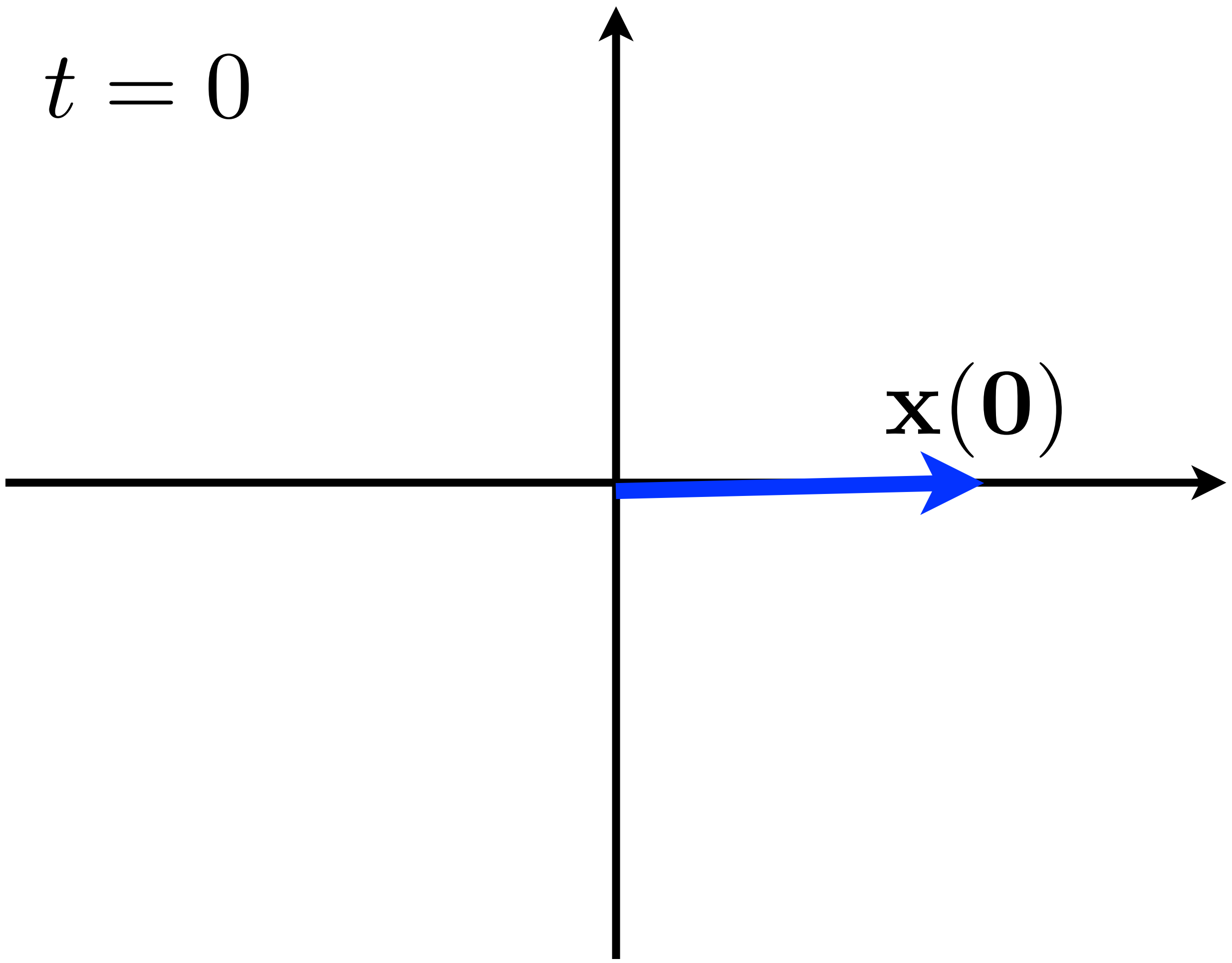
Complex eigenvalues (7.6) - example

$$\mathbf{x(t)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$



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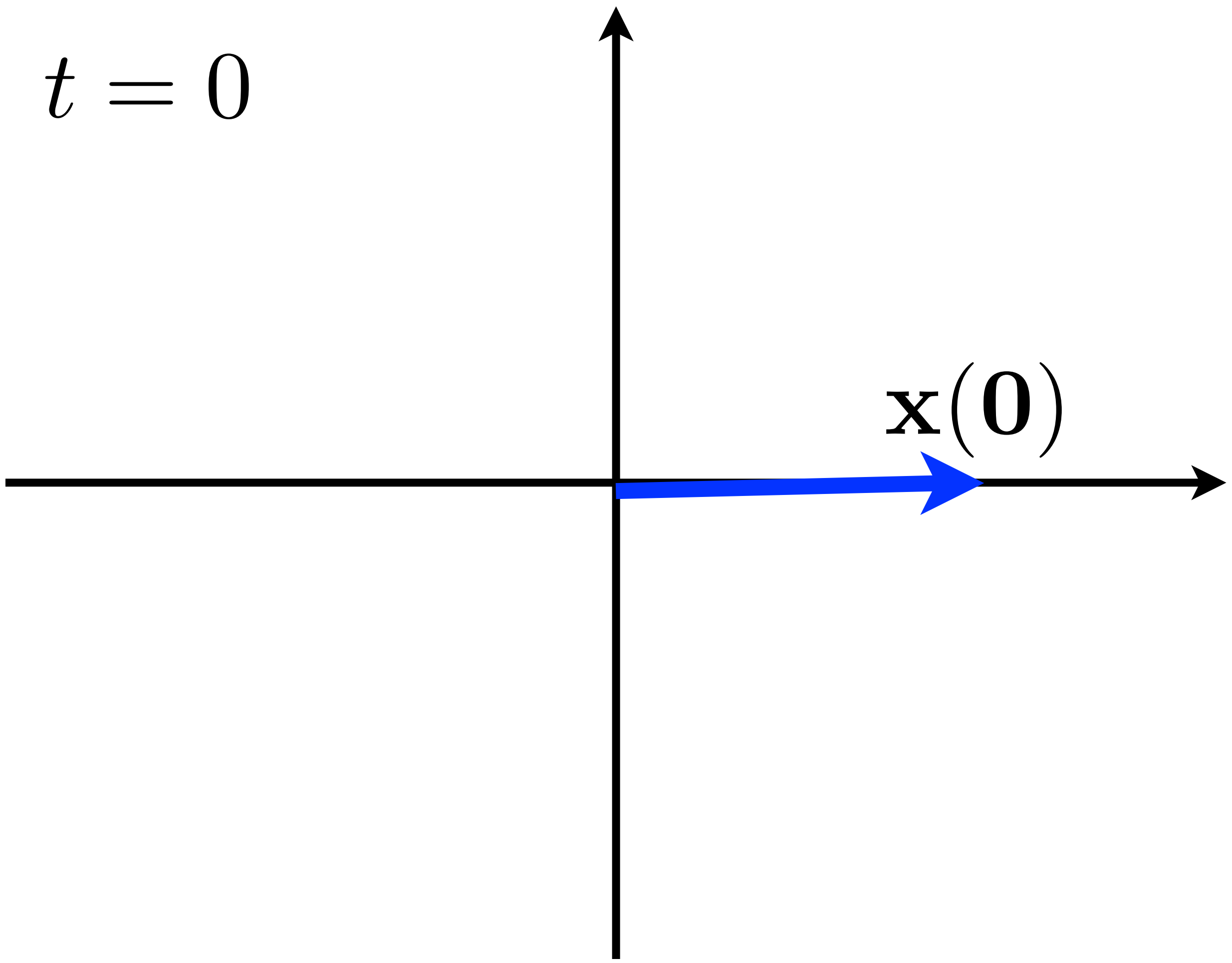

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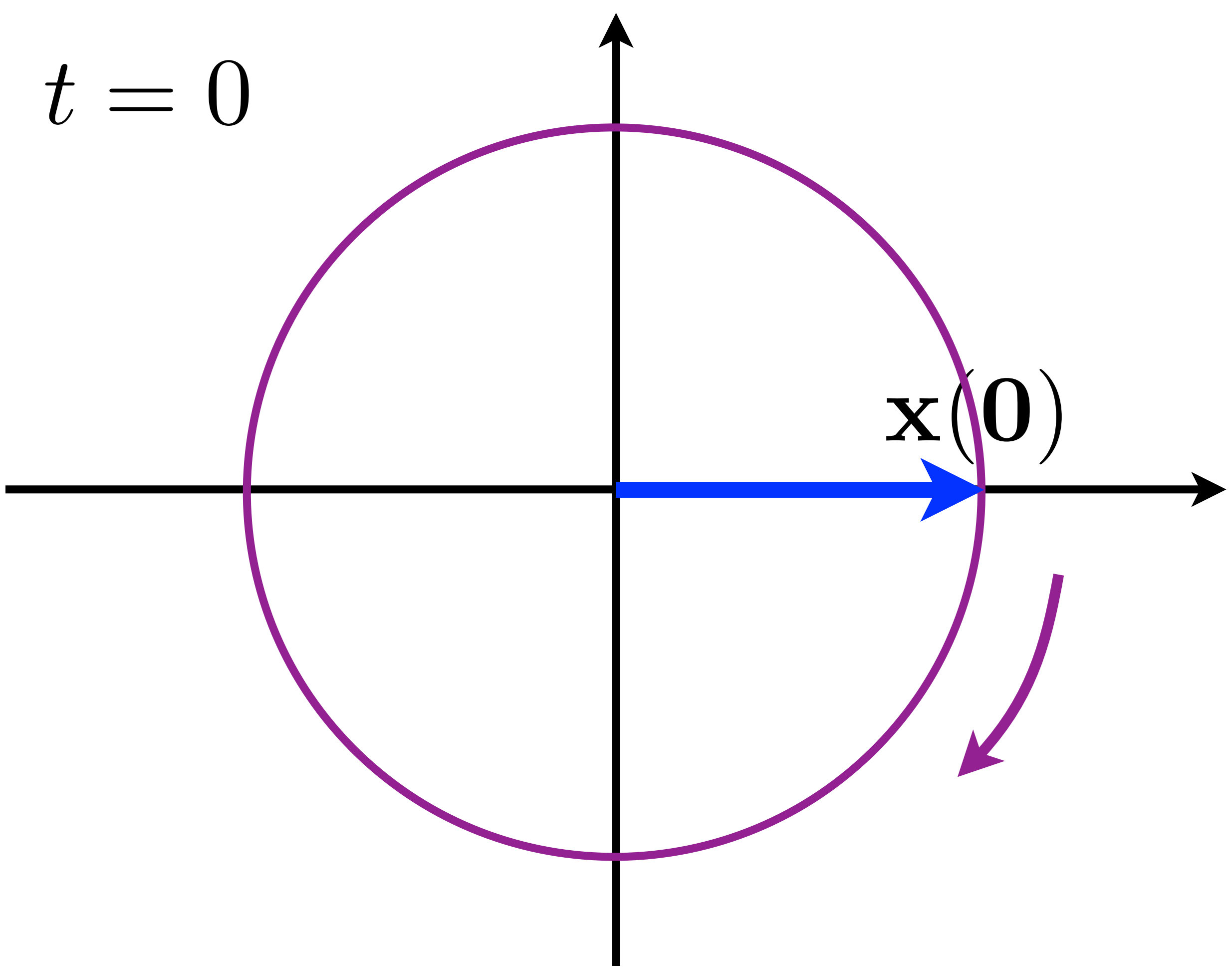

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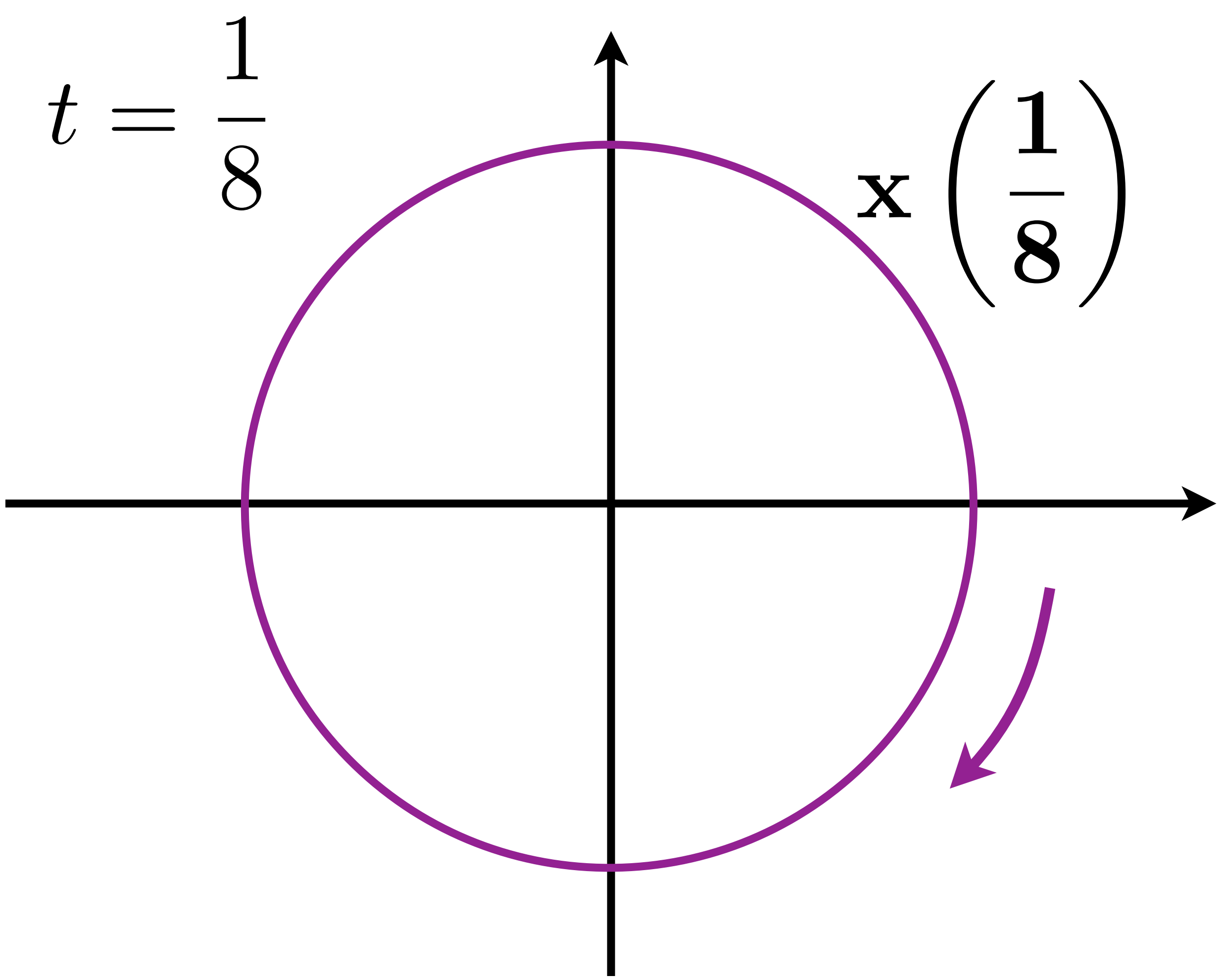
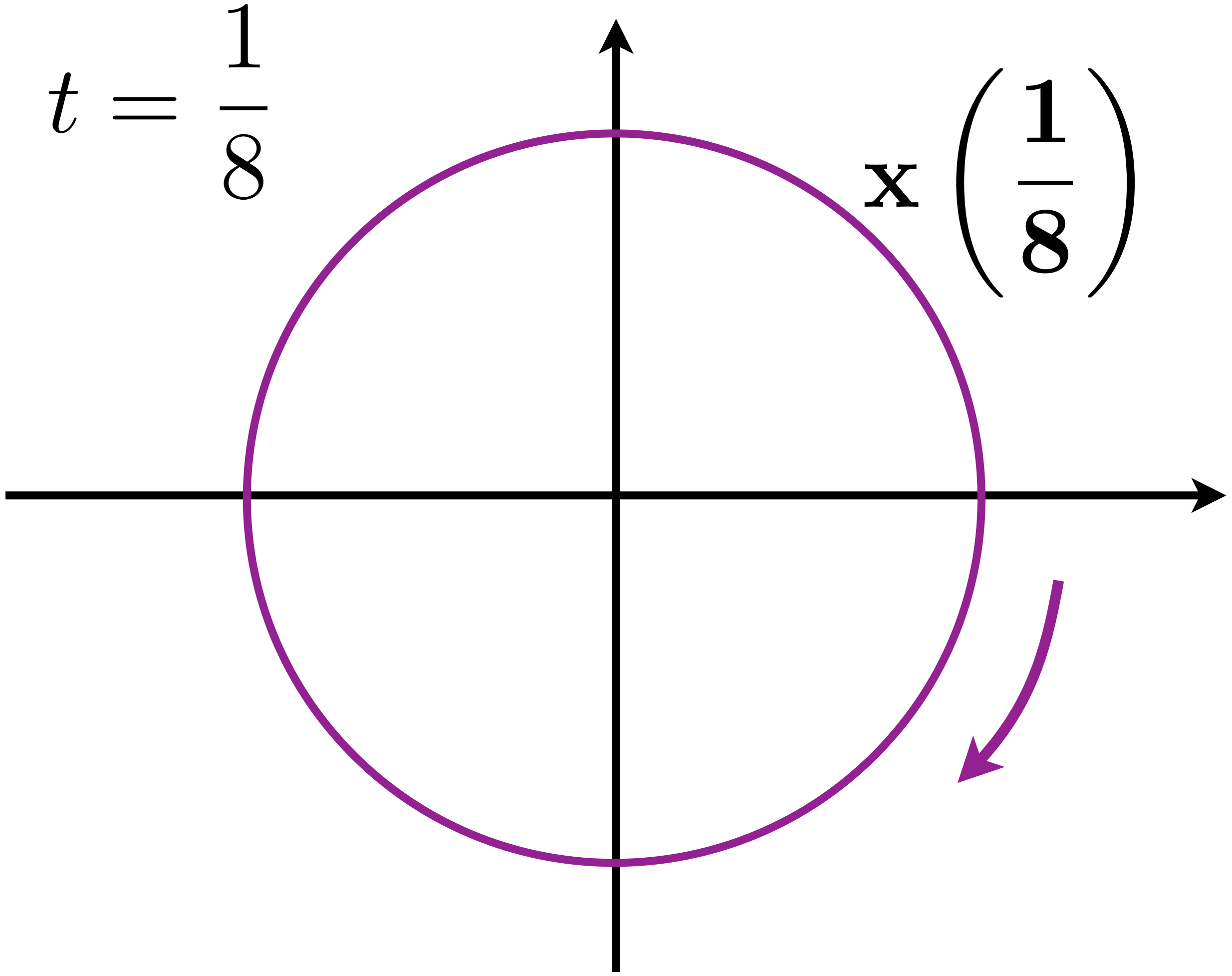


Complex eigenvalues (7.6) - example

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2\pi t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\pi t)$$

$$\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos\left(\frac{\pi}{4}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin\left(\frac{\pi}{4}\right)$$

$$t = \frac{1}{8}$$



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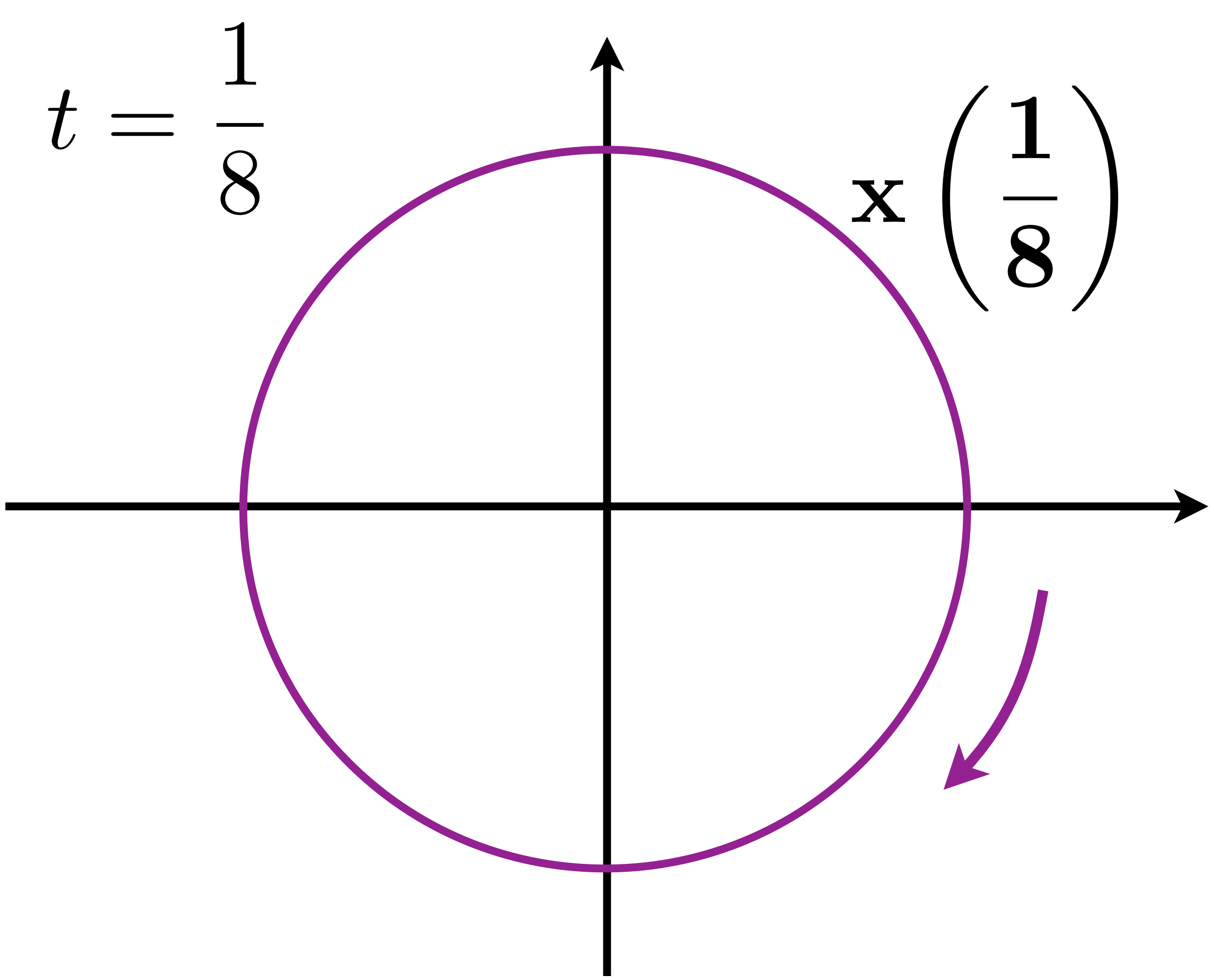

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$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

Complex eigenvalues (7.6) - example

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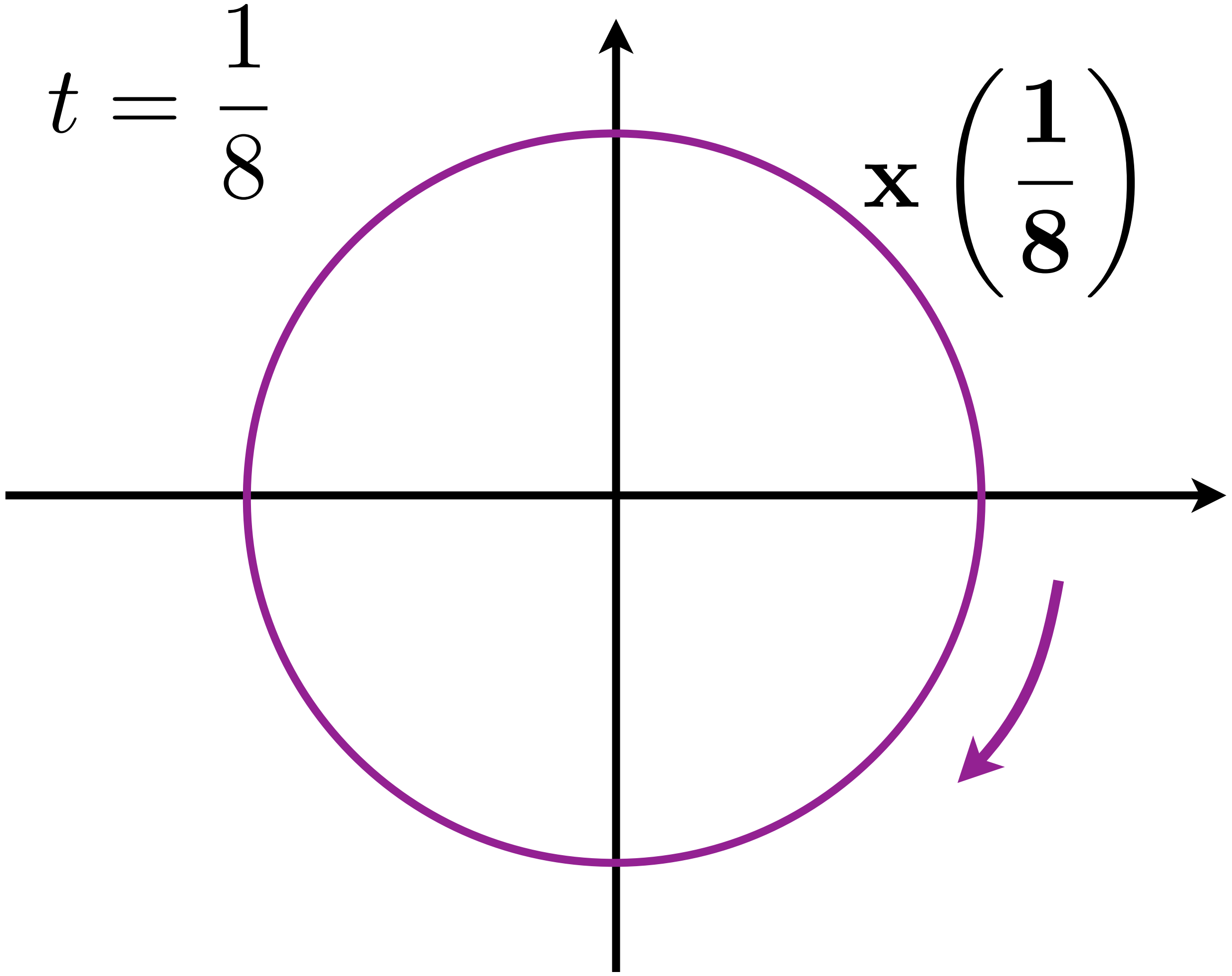

$$\begin{aligned}\mathbf{x}\left(\frac{\mathbf{1}}{\mathbf{8}}\right) &= \begin{pmatrix}1\\0\end{pmatrix}\cos\left(\frac{\pi}{4}\right) + \begin{pmatrix}0\\1\end{pmatrix}\sin\left(\frac{\pi}{4}\right)\\ &= \begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2}} + \begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\\ &= \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\end{aligned}$$

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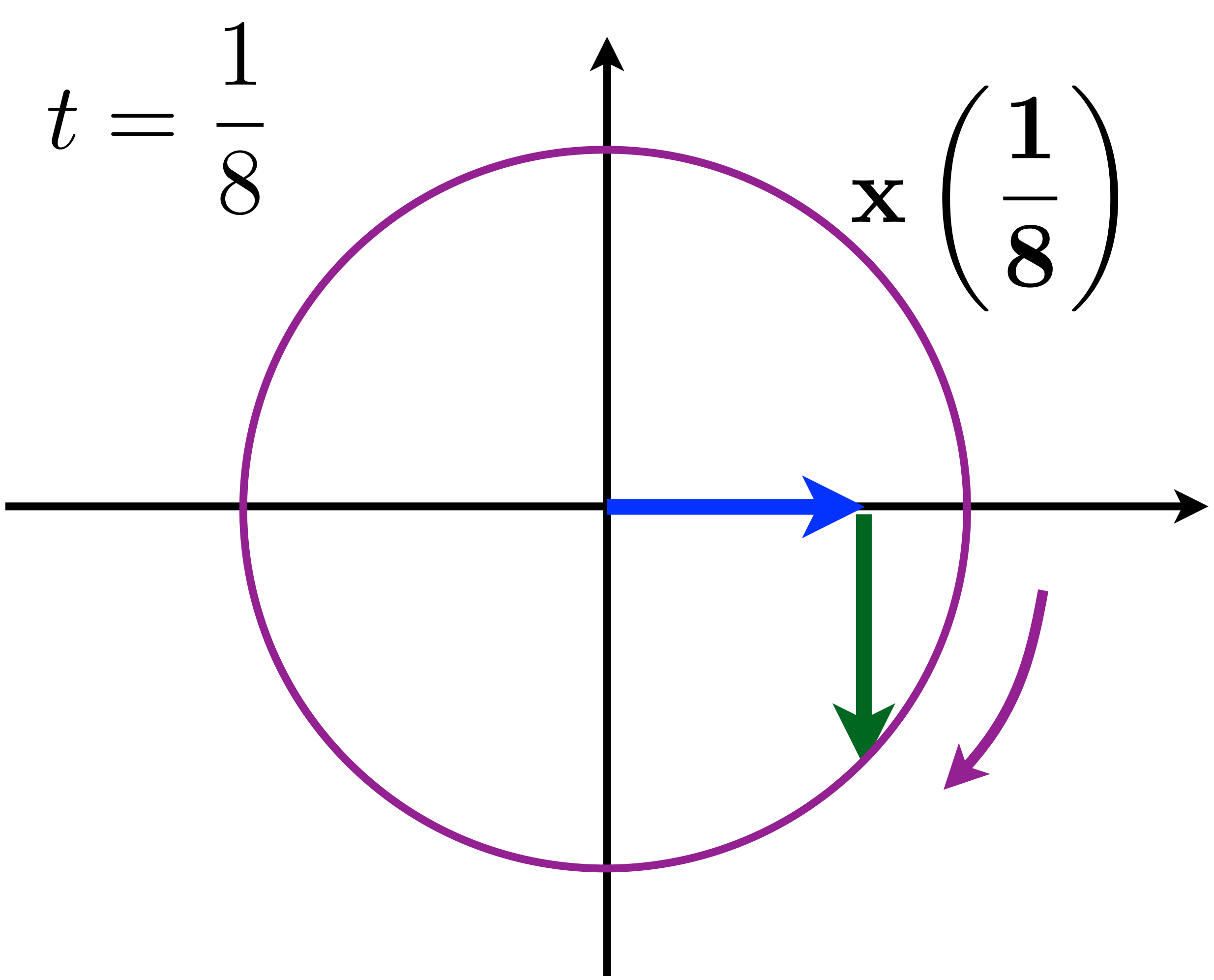

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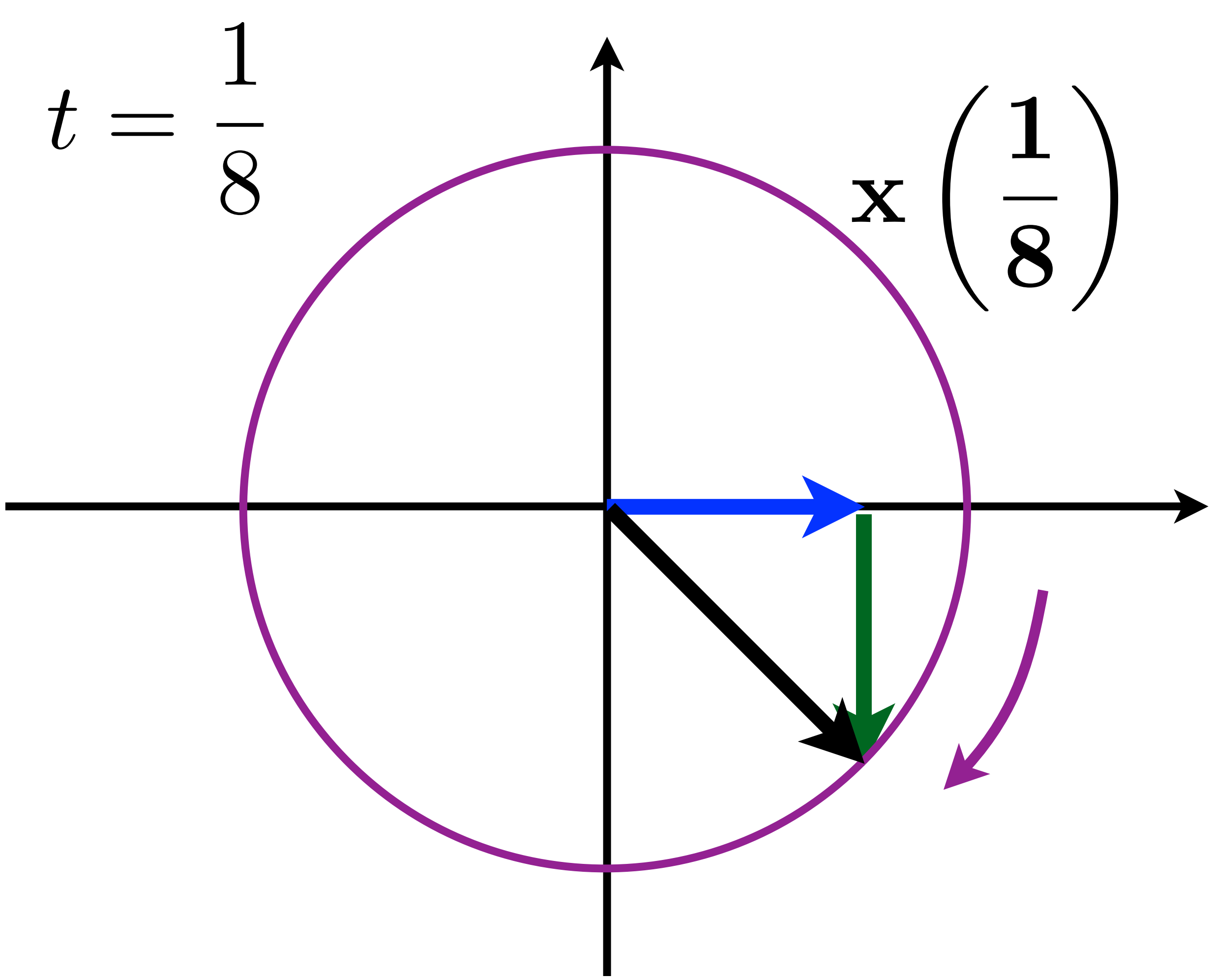

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Complex eigenvalues (7.6) - example

- Same equation, initial condition chosen so that $C_1=0$ and $C_2=1$.

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2\pi t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\pi t)$$

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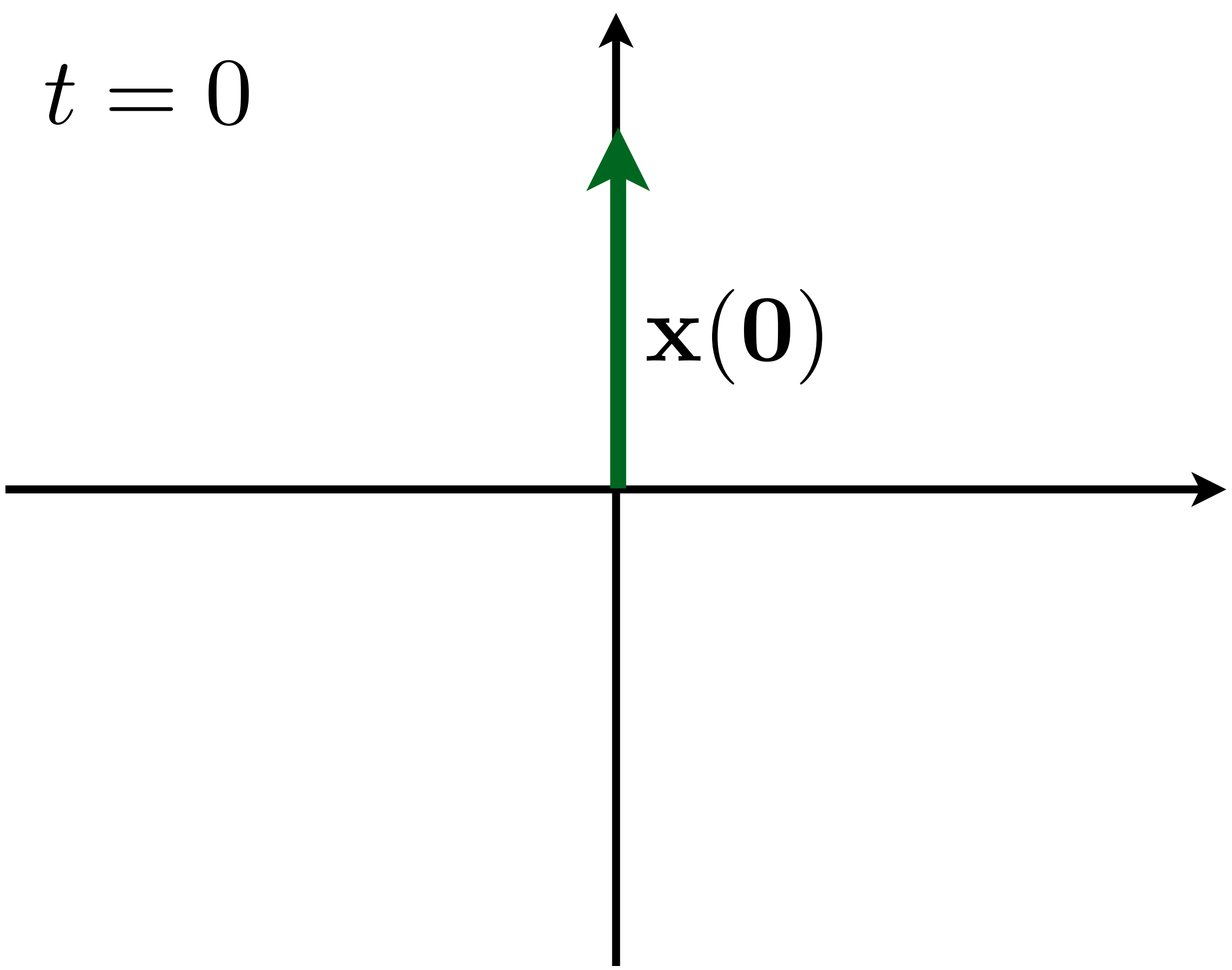

- What happens as t increases?
  - (A) The vector rotates clockwise.
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  - (E) Explain please.

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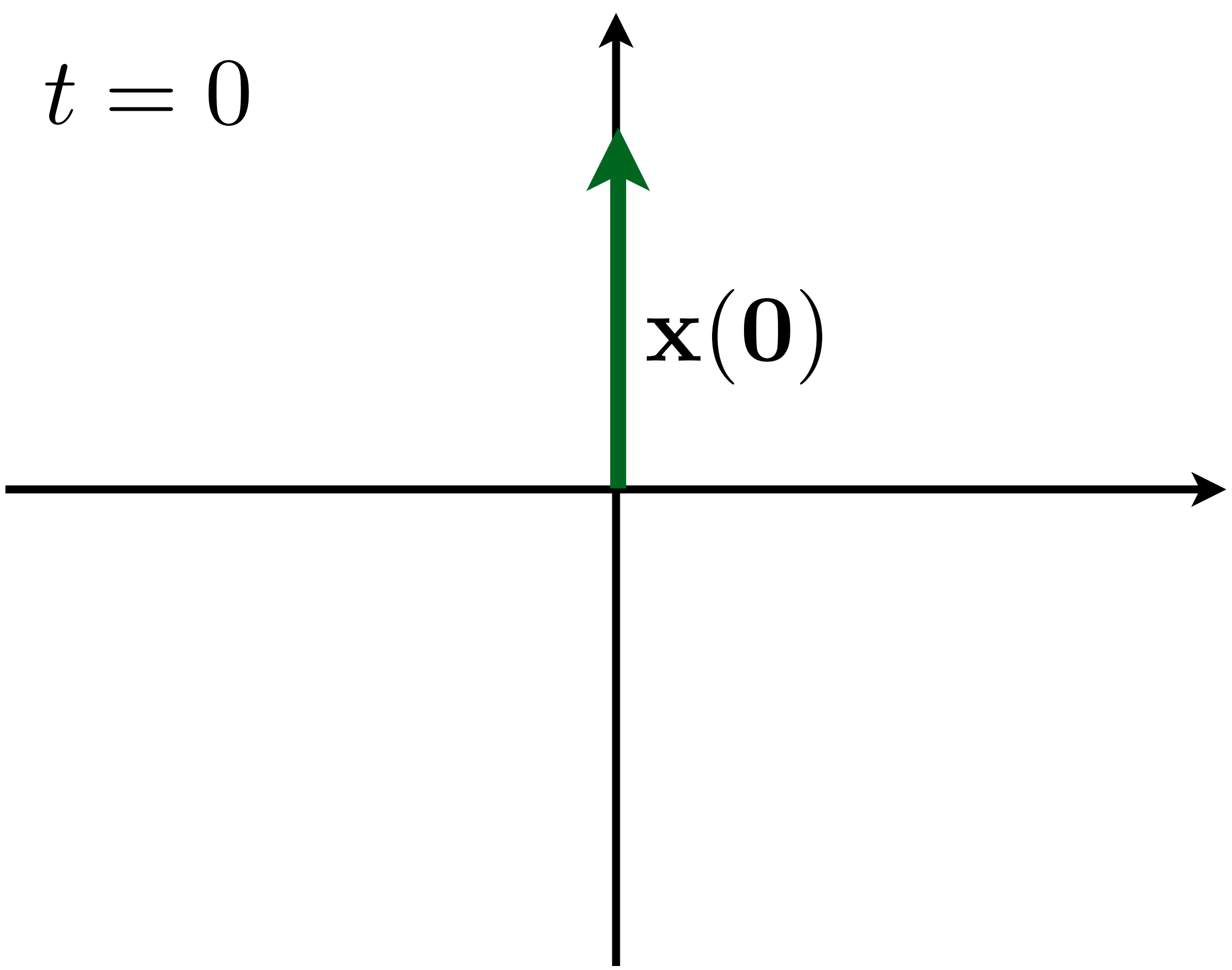

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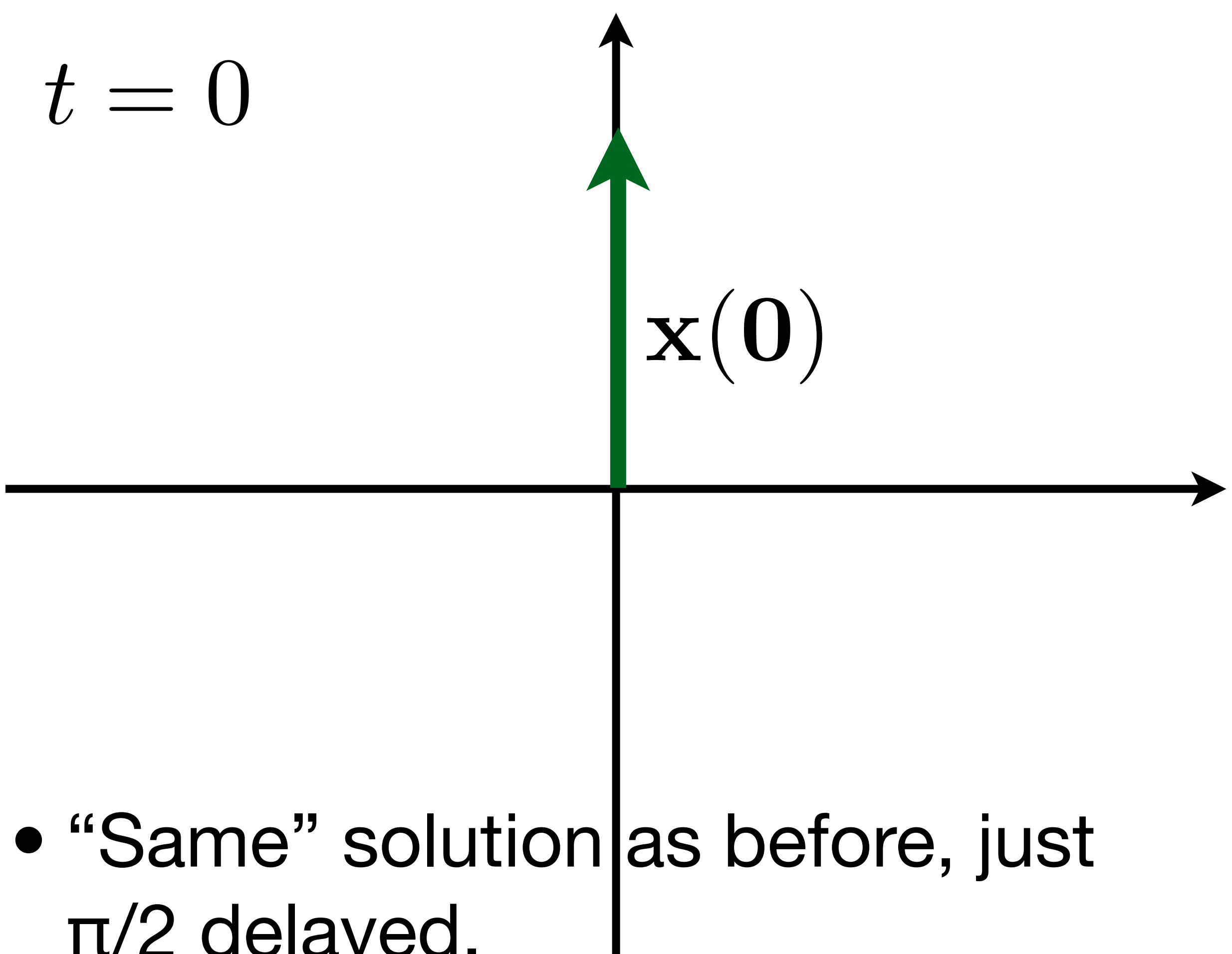

- “Same” solution as before, just π/2 delayed.

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(A) The vector rotates clockwise.

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Complex eigenvalues (7.6) - general case

- Looking at the general solution again...

$$\mathbf{x}(\mathbf{t}) = e^{\alpha t}\left[C_1\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right)\right.$$
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Complex eigenvalues (7.6) - general case

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- Both parts rotate in the exact same way but the C_2 part is delayed by a quarter phase.

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- If an initial condition lies neither parallel to vector **a** nor to vector **b**, C_1 and C2 allow for intermediate phases to be achieved.

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- If an initial condition lies neither parallel to vector **a** nor to vector **b**, C_1 and C2 allow for intermediate phases to be achieved.

- **x(t)** can be rewritten (using trig identities) as

$$\mathbf{x}(\mathbf{t}) = Me^{\alpha t}\left(\mathbf{a}\cos(\beta t - \phi) - \mathbf{b}\sin(\beta t - \phi)\right)$$

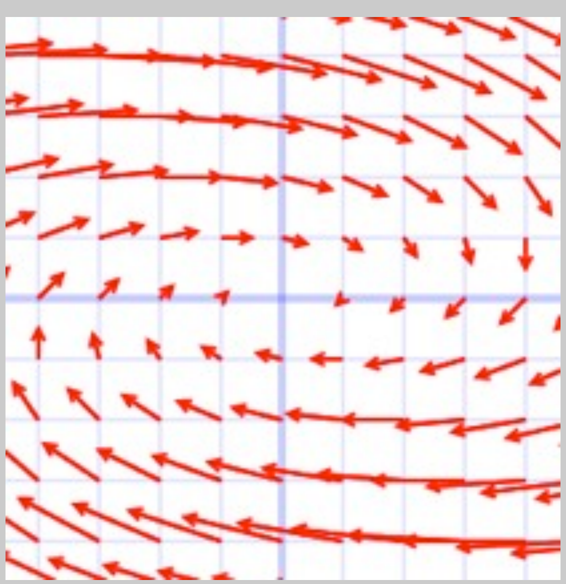
where M and ϕ are constants to replace C_1 and C_2.

Complex eigenvalues (7.6) - example

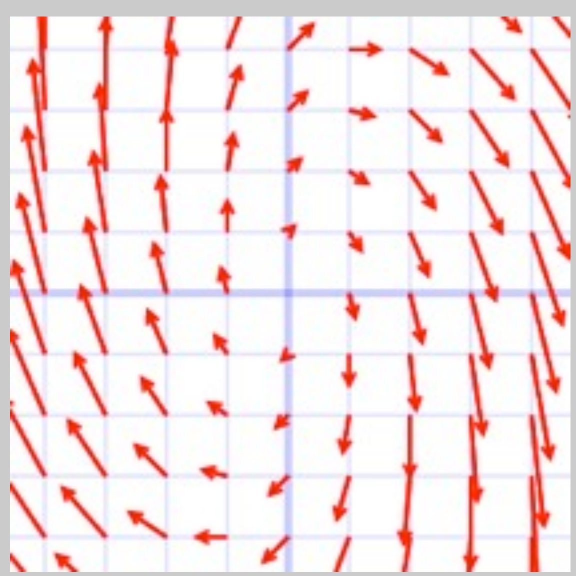
- Back to our earlier example where we found the general solution

$$\mathbf{x}(\mathbf{t}) = e^{t}\left(C_1\left(\begin{pmatrix}1\\0\end{pmatrix}\cos(2t) - \begin{pmatrix}0\\2\end{pmatrix}\sin(2t)\right) + C_2\left(\begin{pmatrix}1\\0\end{pmatrix}\sin(2t) + \begin{pmatrix}0\\2\end{pmatrix}\cos(2t)\right)\right)$$

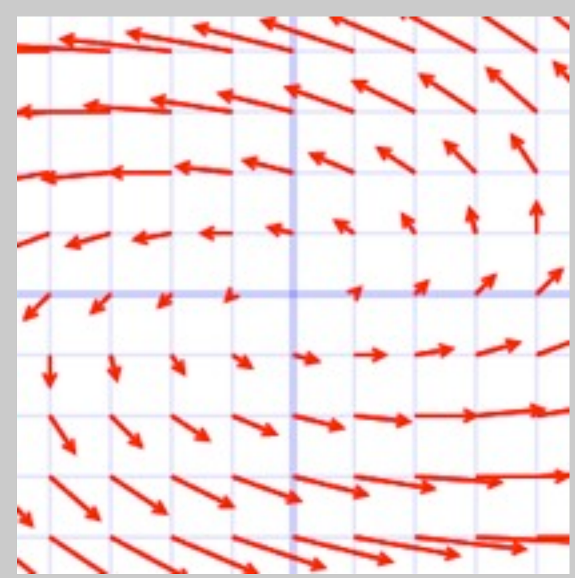
(A)



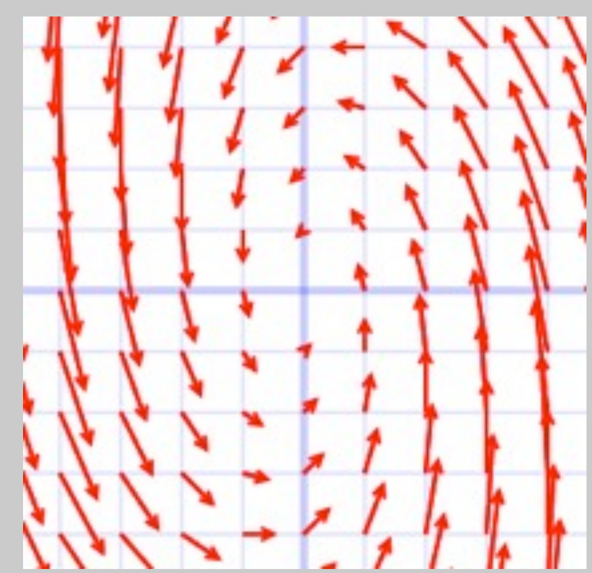
(B)



(C)



(D)



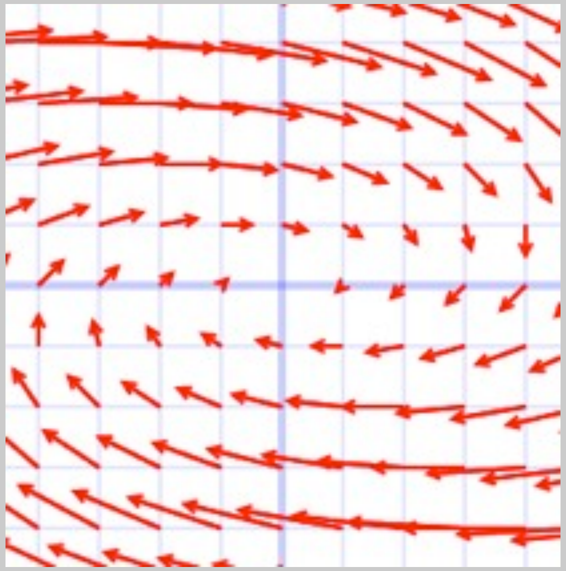
(E) Explain, please.

Complex eigenvalues (7.6) - example

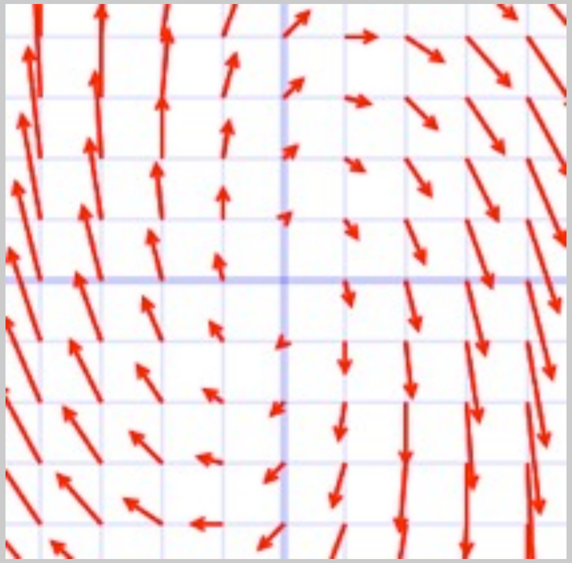
- Back to our earlier example where we found the general solution

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$

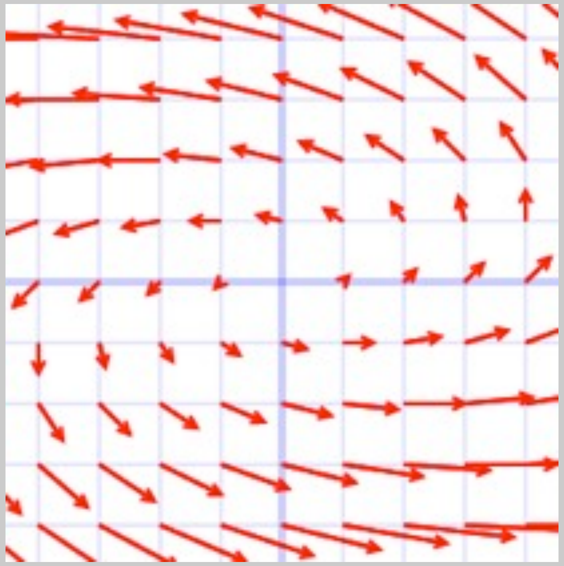
(A)



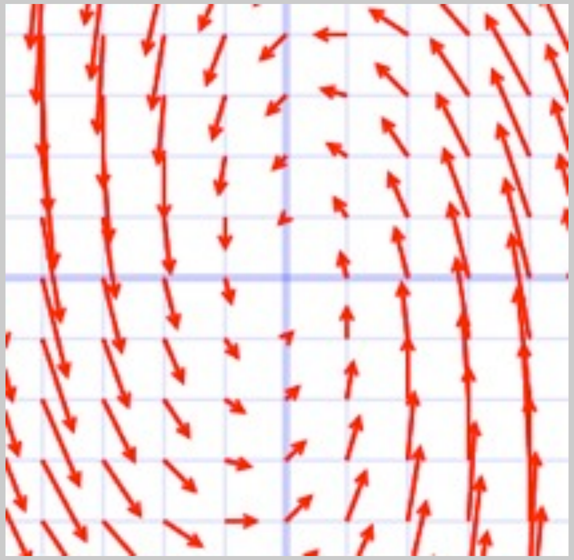
(B) ☆



(C)



(D)

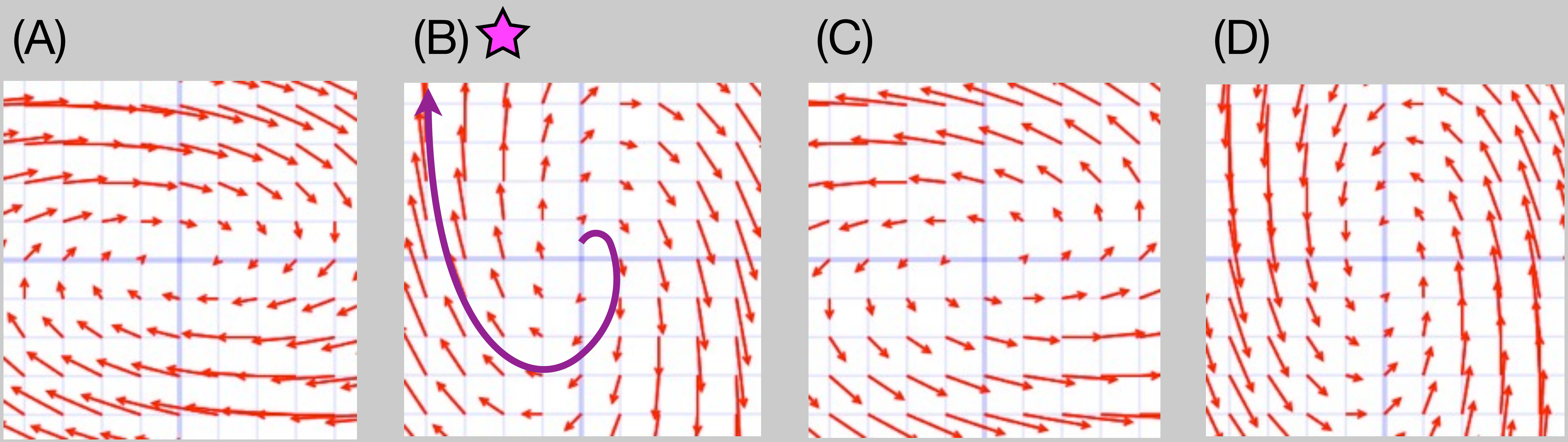


(E) Explain, please.

Complex eigenvalues (7.6) - example

- Back to our earlier example where we found the general solution

$$\mathbf{x}(\mathbf{t}) = e^t \left(C_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(2t) \right) + C_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(2t) \right) \right)$$



(E) Explain, please.