Today

- Homework
  - one WW problem to appear today
  - more TBA from the textbook to be handed in at the start of the tutorial Monday April 7.
- Tutorial on Monday - worksheet instead of quiz.
- Orthogonality of sine and cosine functions
- Fourier series approximations to functions
- Using Fourier series to solve the Diffusion Equation

Solving initial conditions using linear algebra

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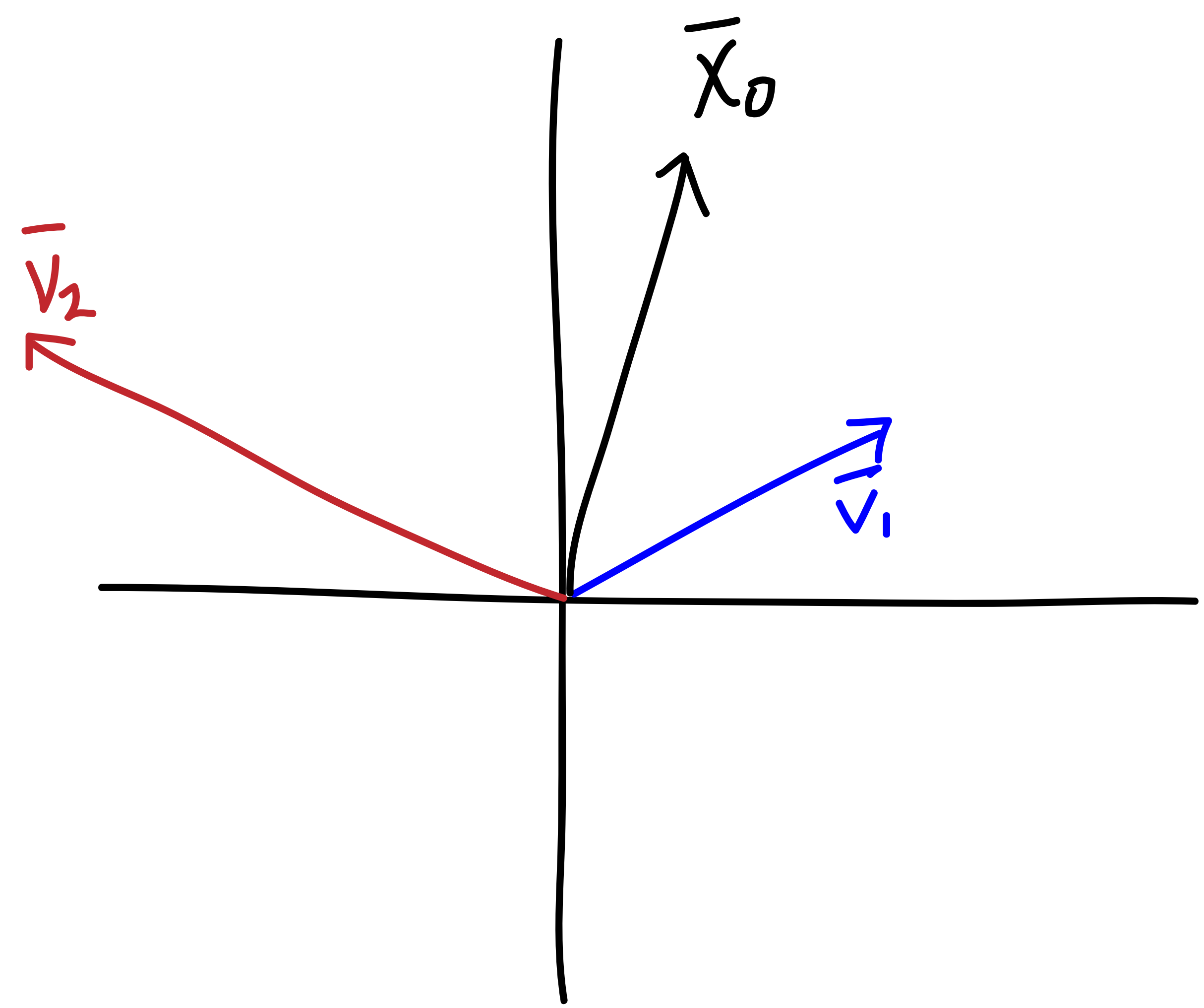

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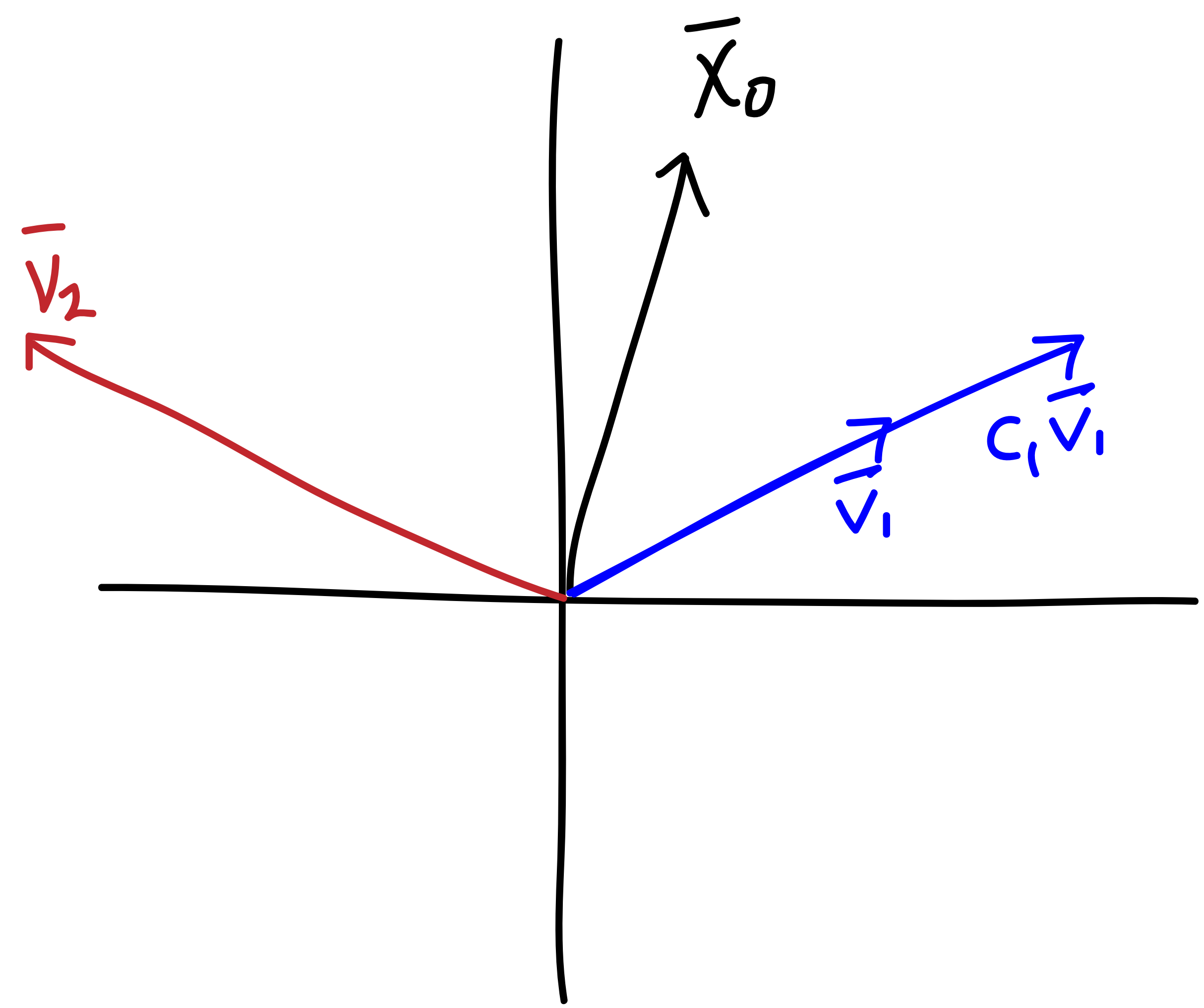

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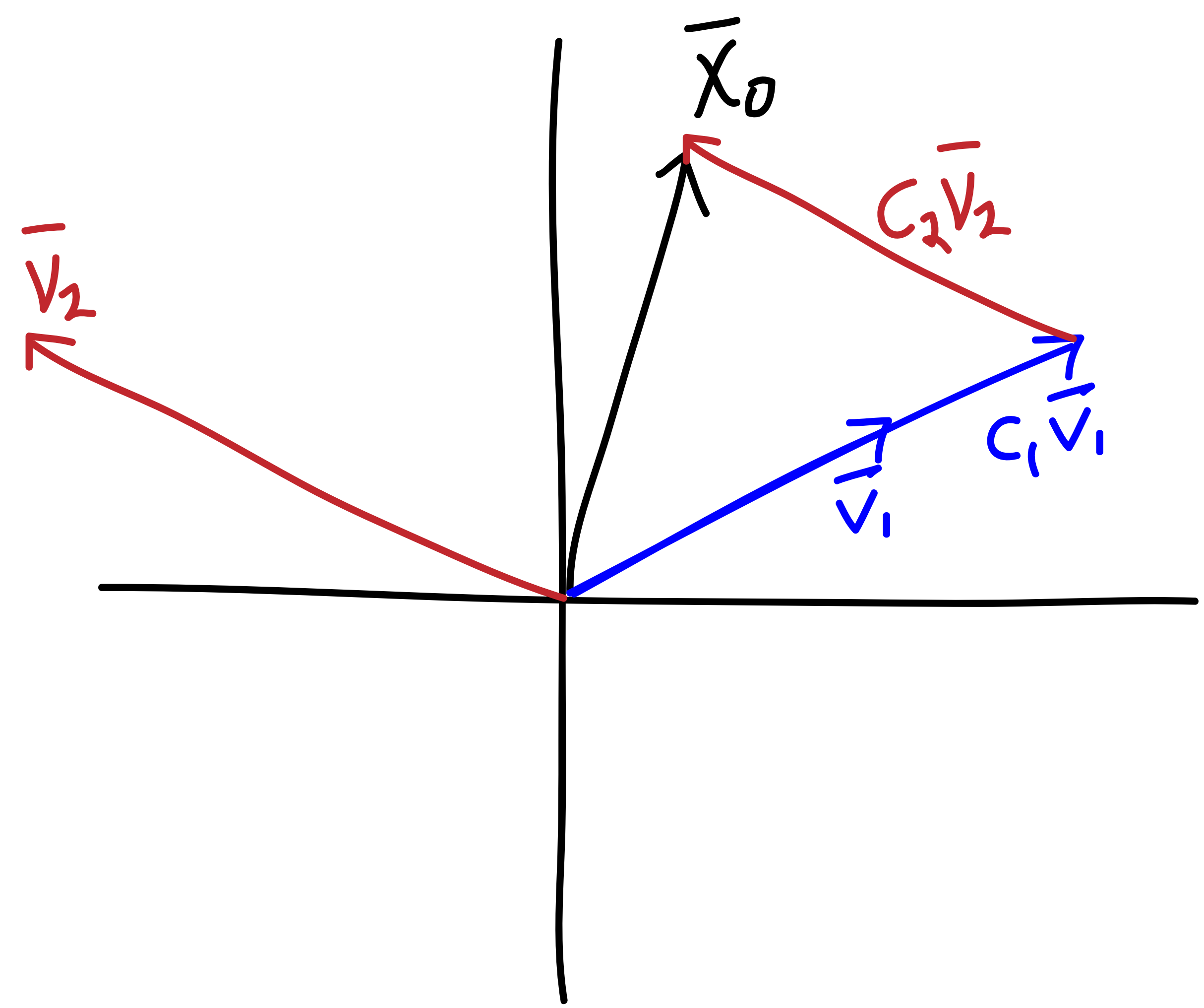

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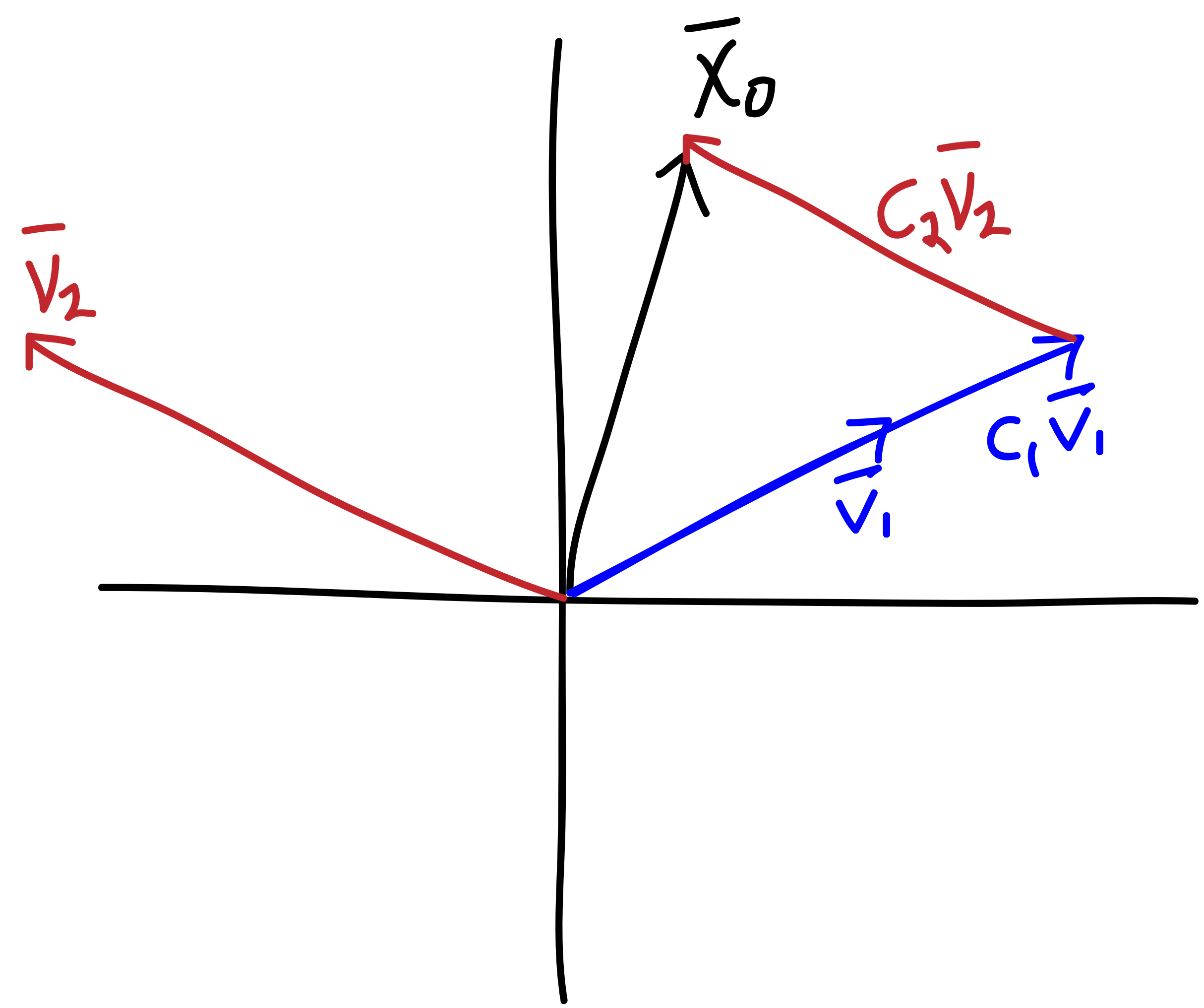

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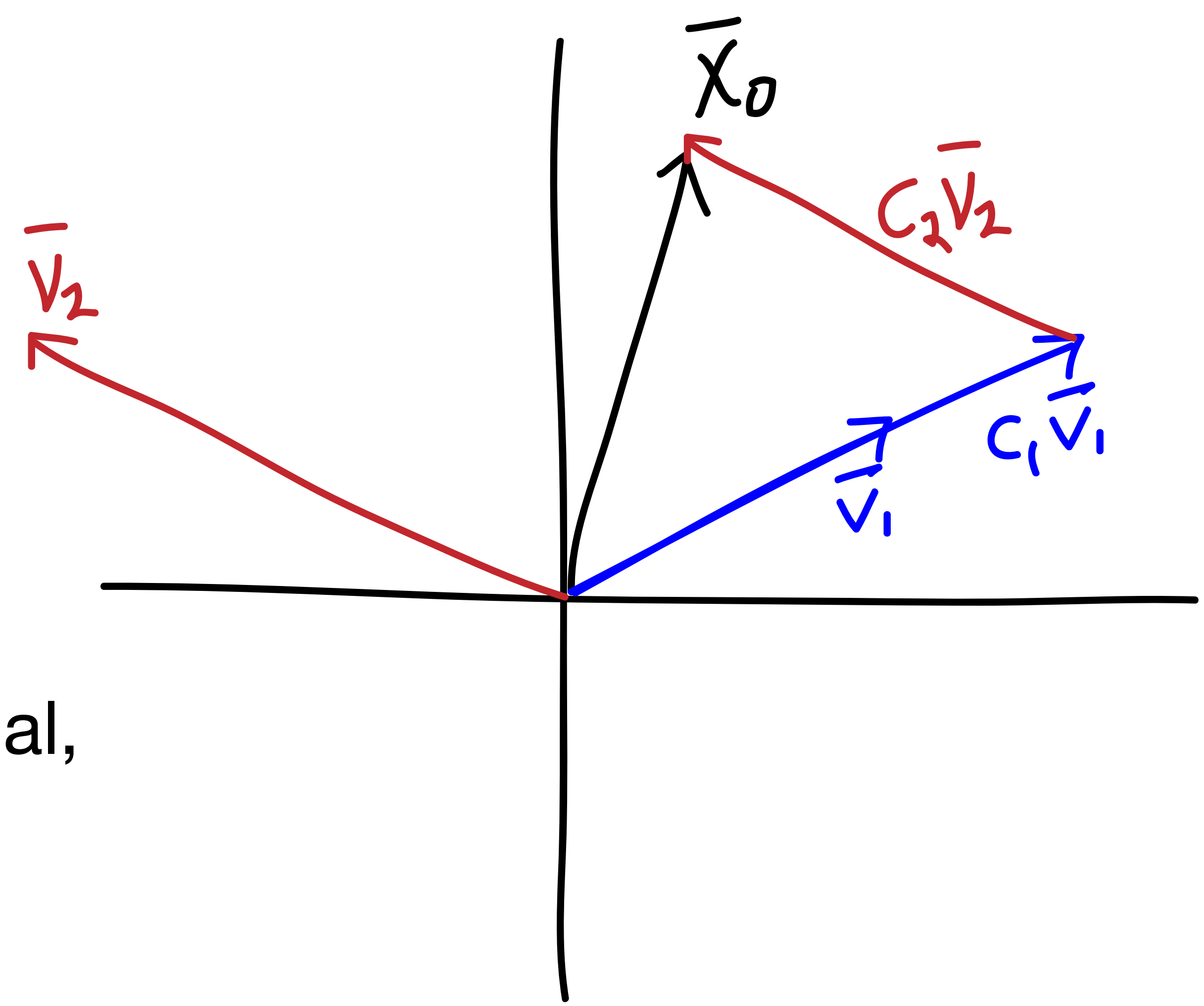

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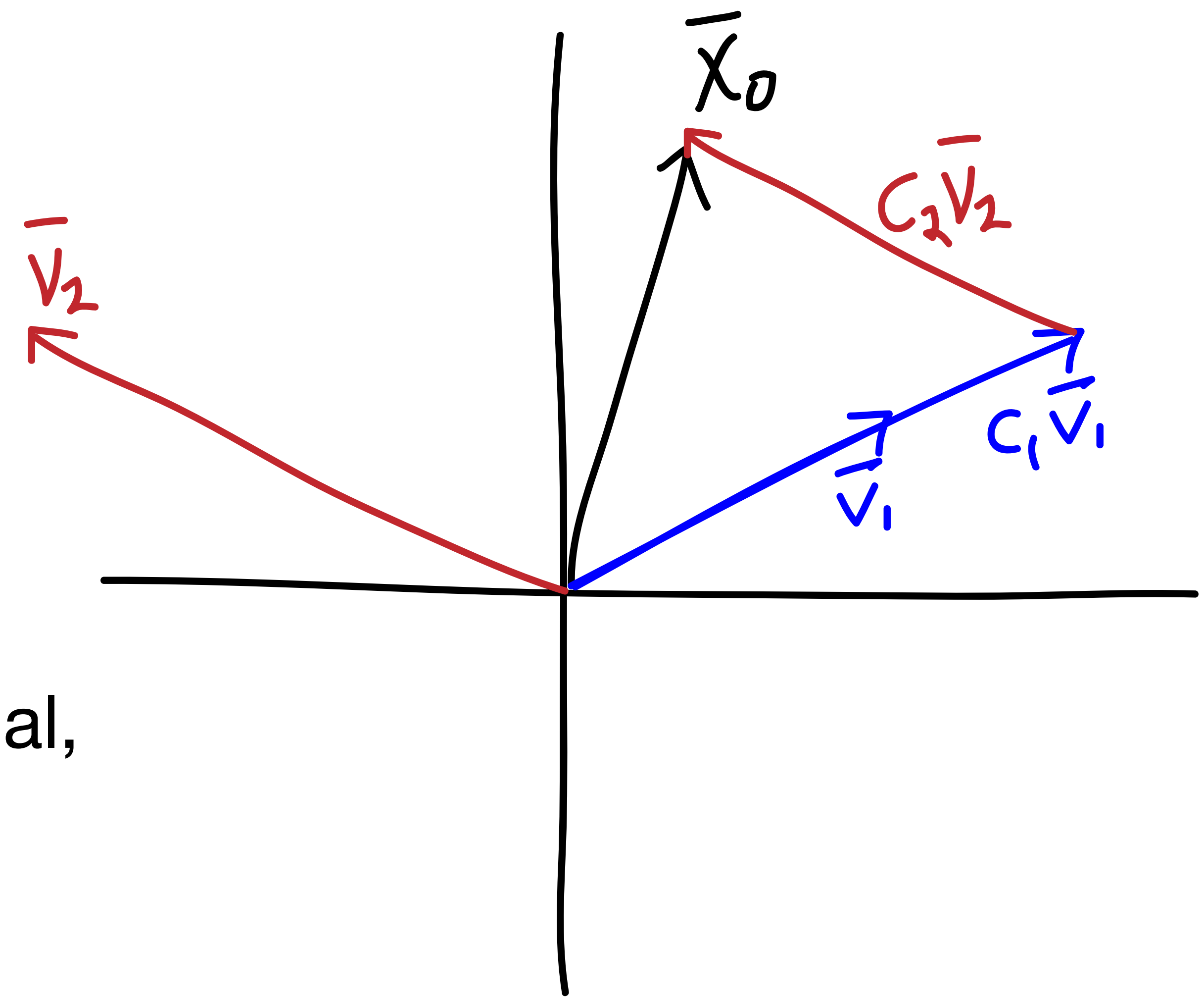

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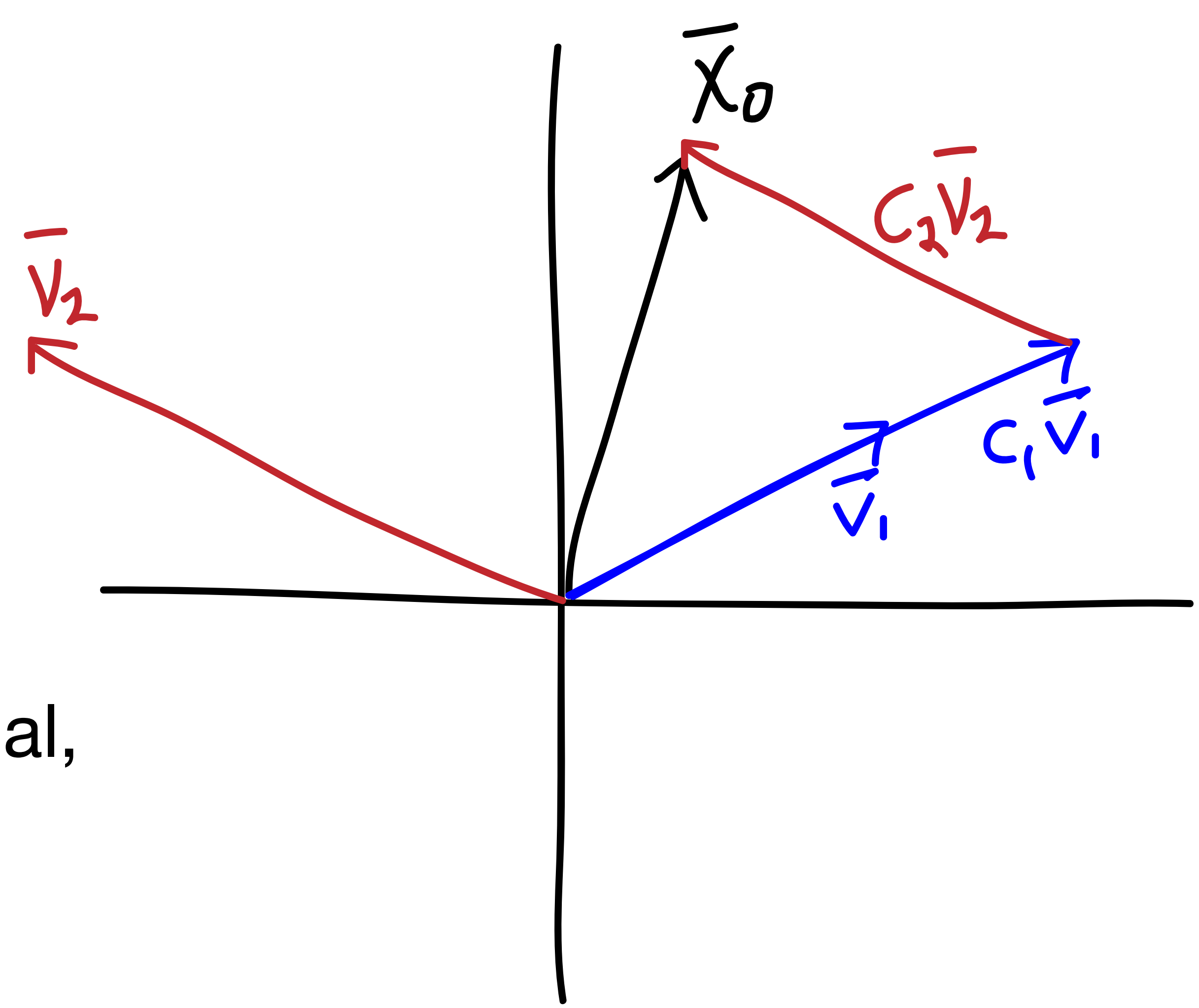

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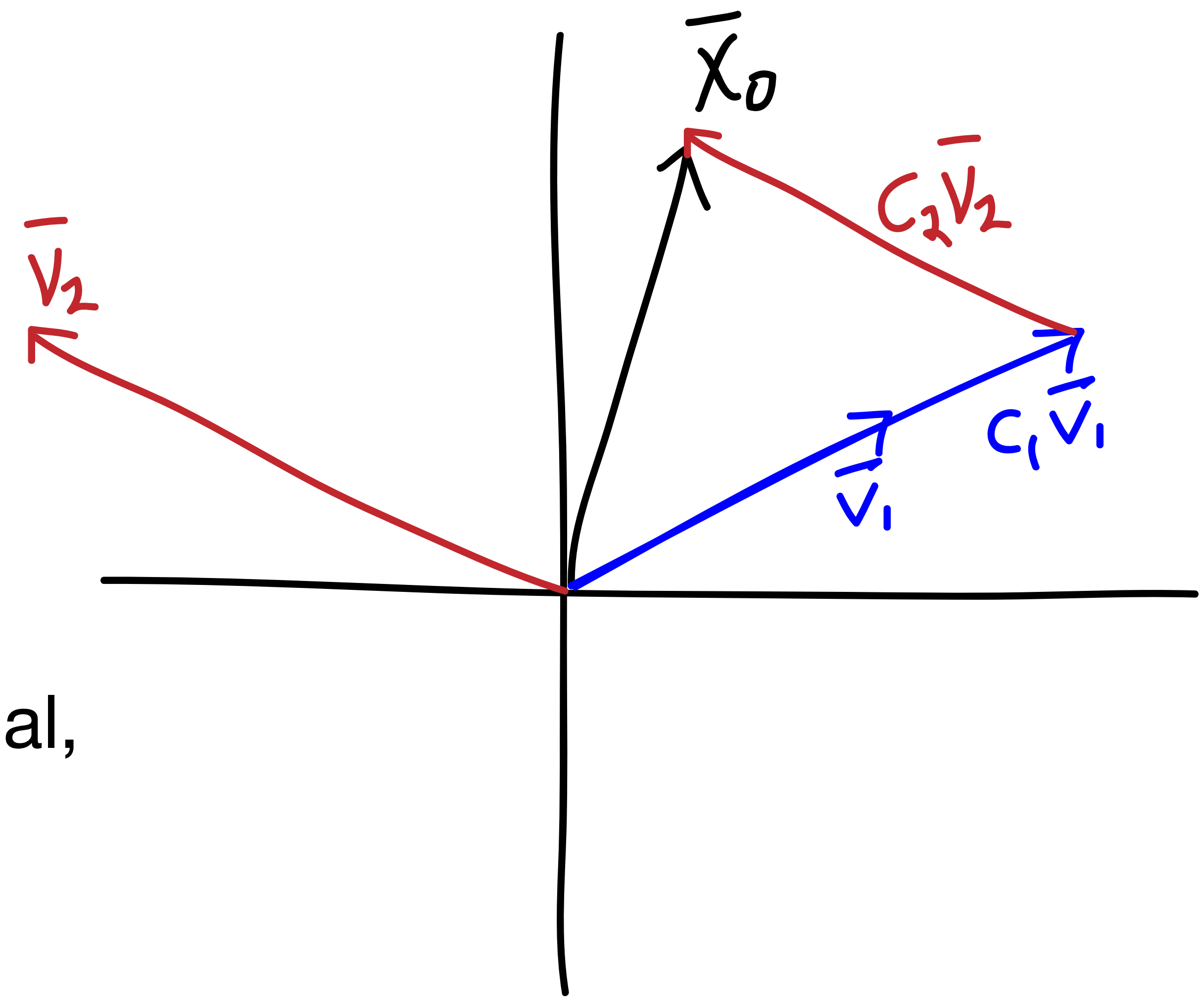

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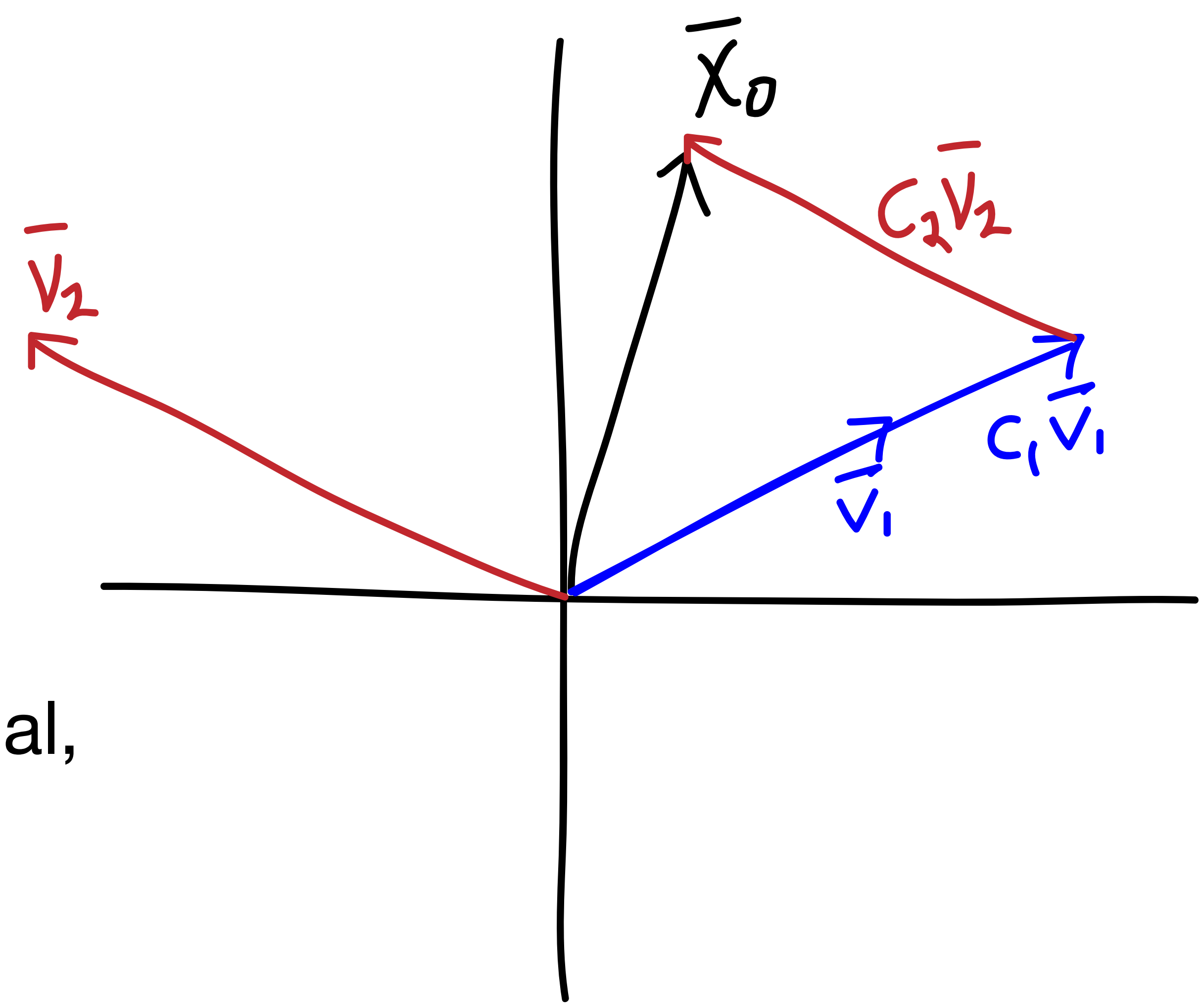

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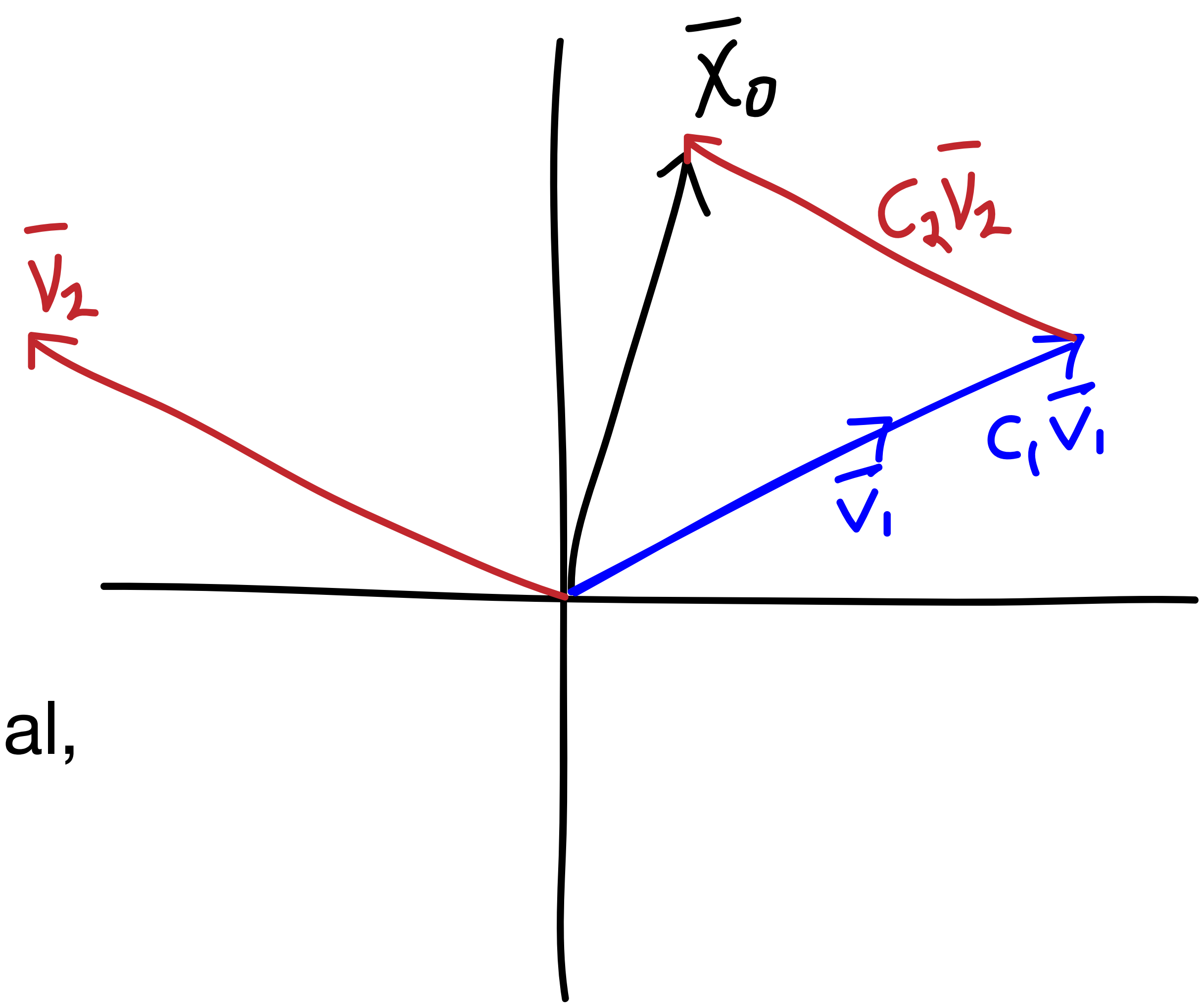

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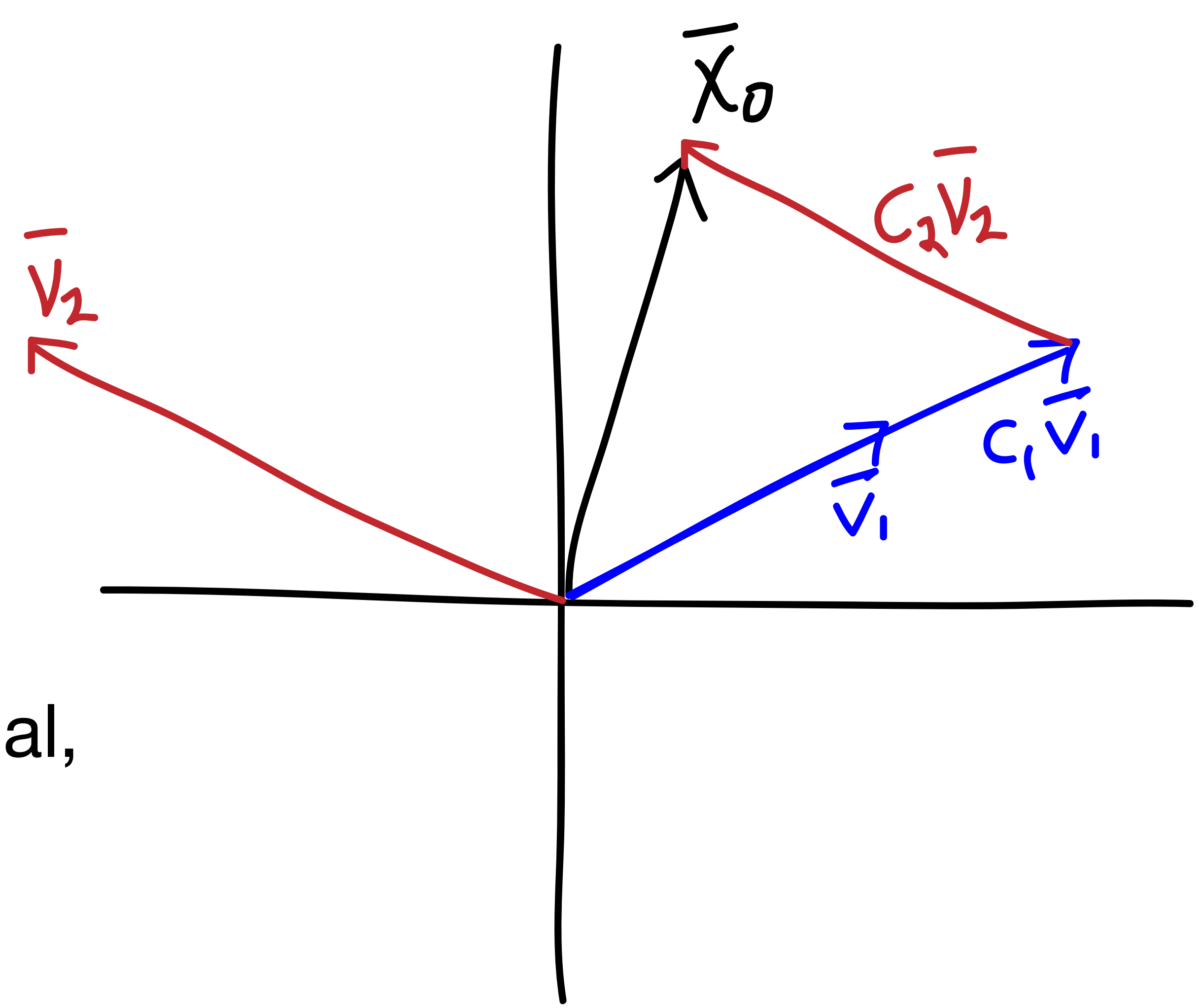

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- For the Diffusion Equation, we found that to solve the problem

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2} \qquad \begin{aligned} c(L,t) &= 0 \\ c(0,t) &= 0 \end{aligned} \qquad c(x,0) = f(x)$$

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(A) 0

(B) π

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Integral of an odd function over a symmetric interval = 0

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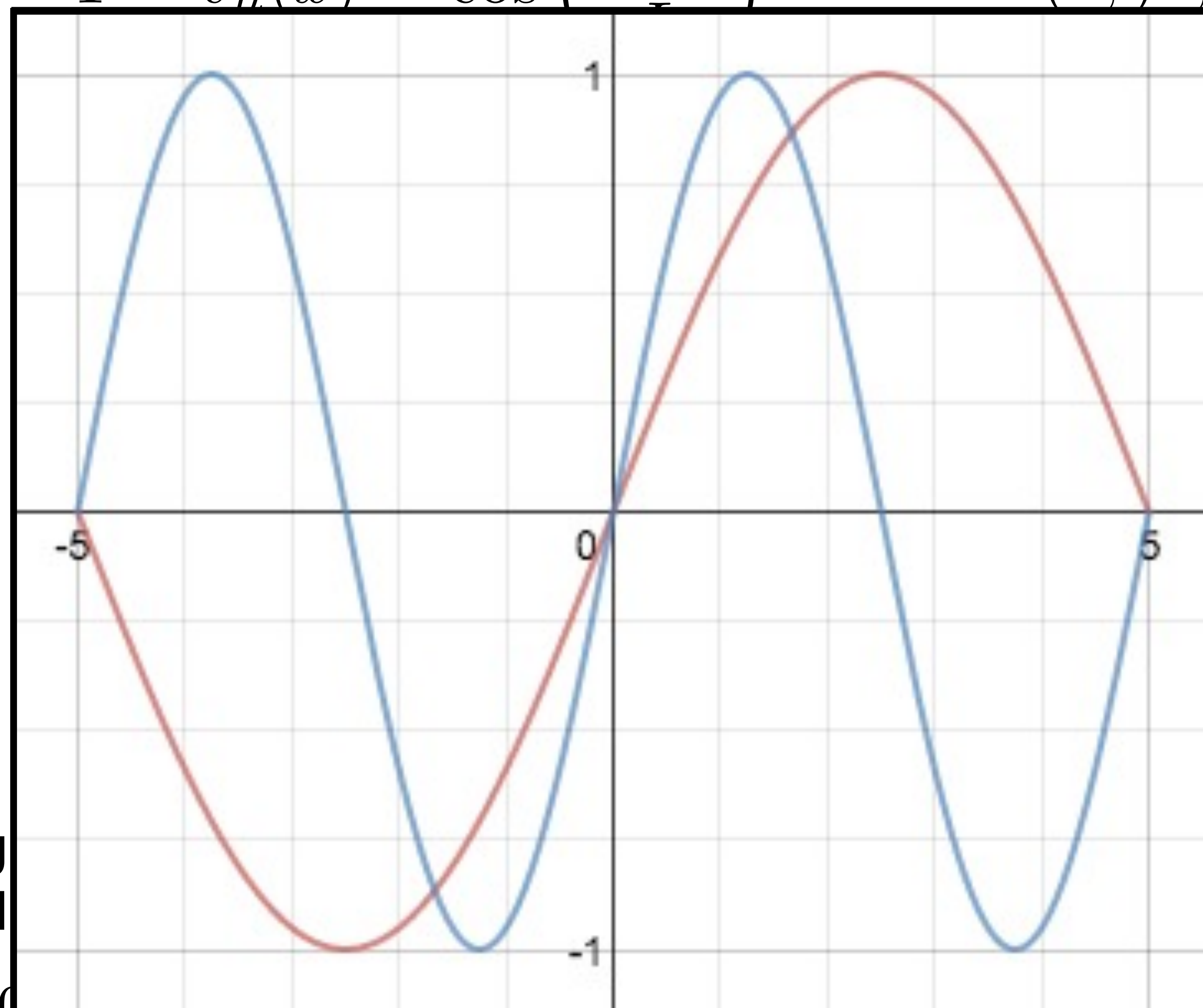

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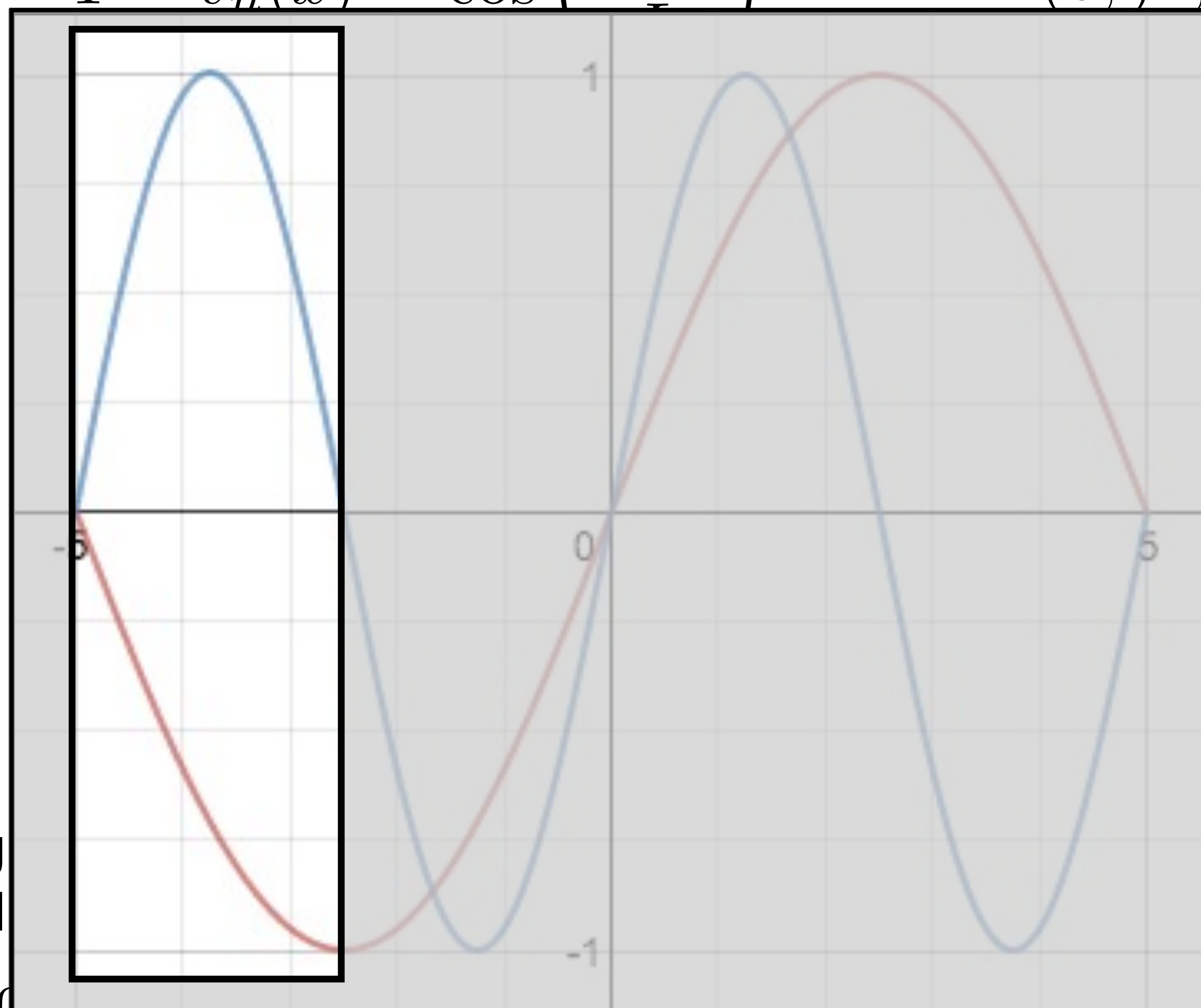

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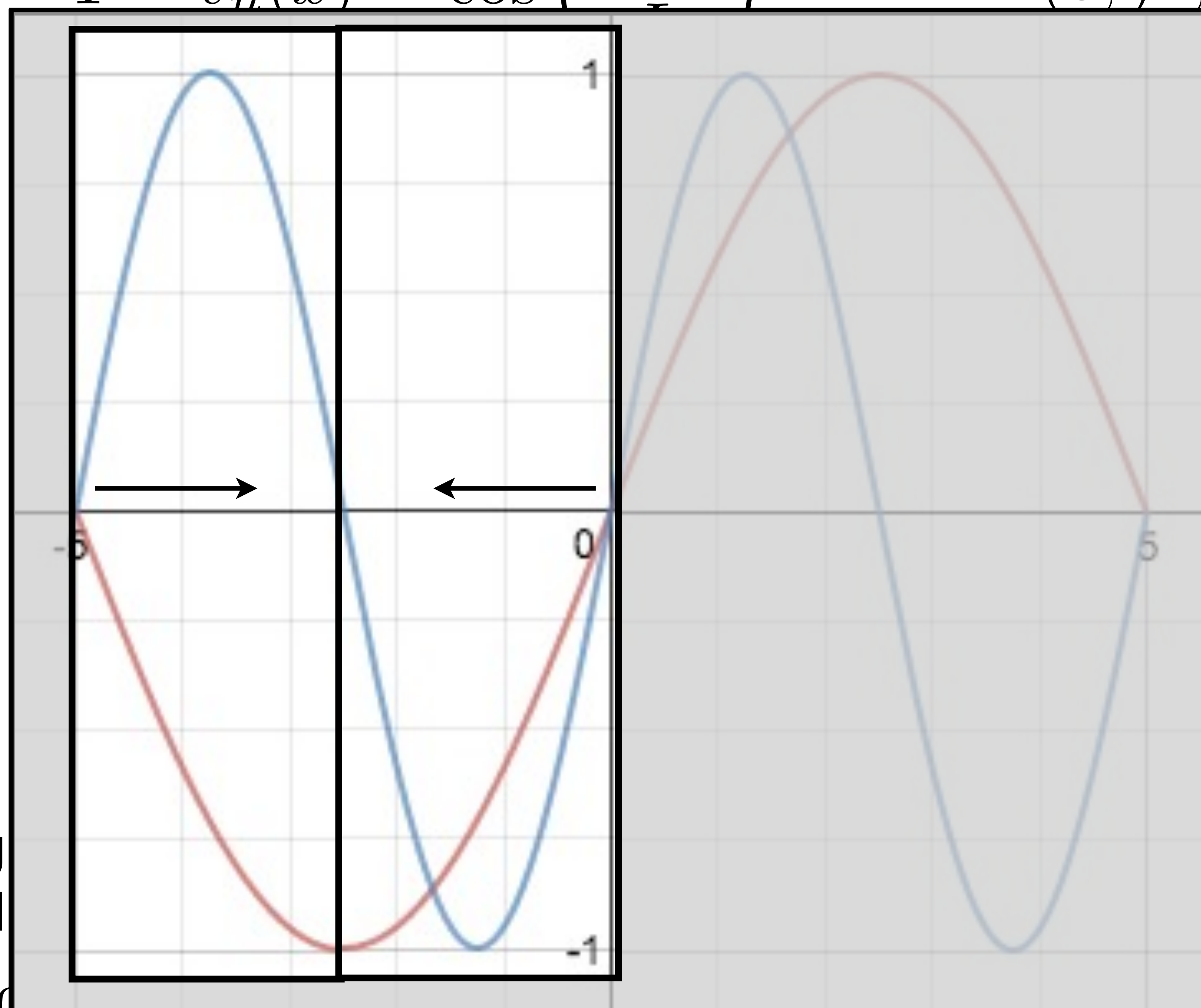

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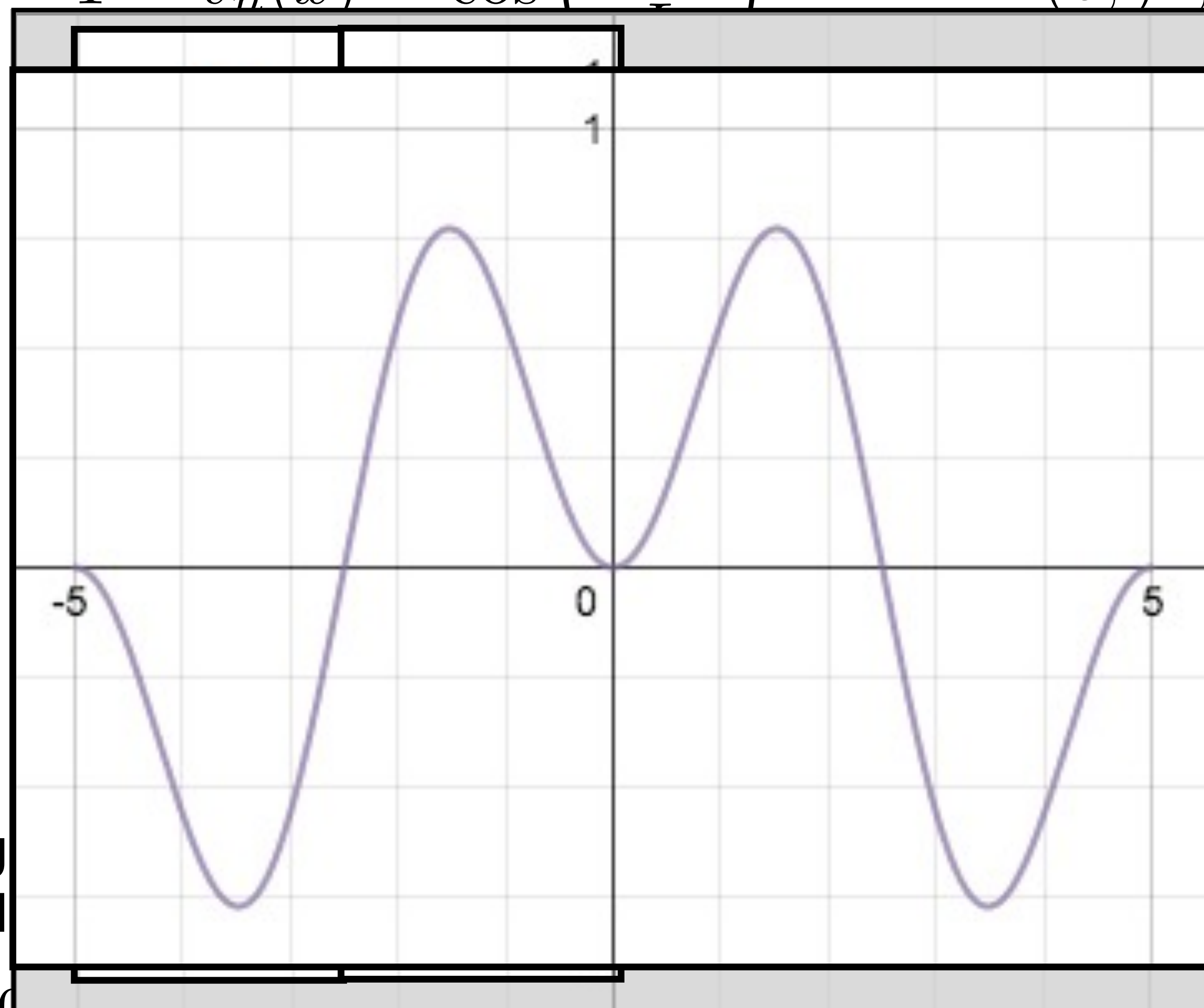

$v_0 \circ v_n =$

$(m \neq n)$

★(A) 0

(B) π

(C) π/2

(D) nπ/2

2

/2

Integral of a trig
over one period

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$$g(x) \circ h(x) = \int_{-L}^{L} g(x)h(x)\ dx$$

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$v_0 \circ v_n =$

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Integral of a trig function over one period = 0

$v_0 \circ w_n = 0$

$v_m \circ w_n =$

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Integral of an odd function over a symmetric interval = 0

$v_m \circ v_n =$ $\quad (m \neq n)$

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(B) π

(C) π/2

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$$g(x) \circ h(x) = \int_{-L}^{L} g(x)h(x)\ dx$$

Solving initial conditions using linear algebra

- The only inner products of eigenfunctions that aren't zero:

$$v_0 \circ v_0 = \int_{-L}^{L} 1 \cdot 1 \; dx = 2L$$

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Fourier series

- Taking a step back from PDEs, let's define what a Fourier series is.

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- Define a function $f_{FS}(x)$ on the interval [-L,L] by choosing coefficients A_0, a_n and b_n and setting

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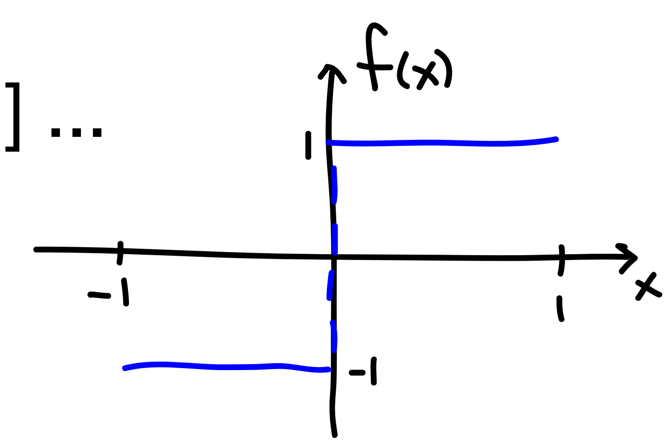
- Let's check for $f(x) = 2u_0(x)-1$ on the interval [-1,1] ...



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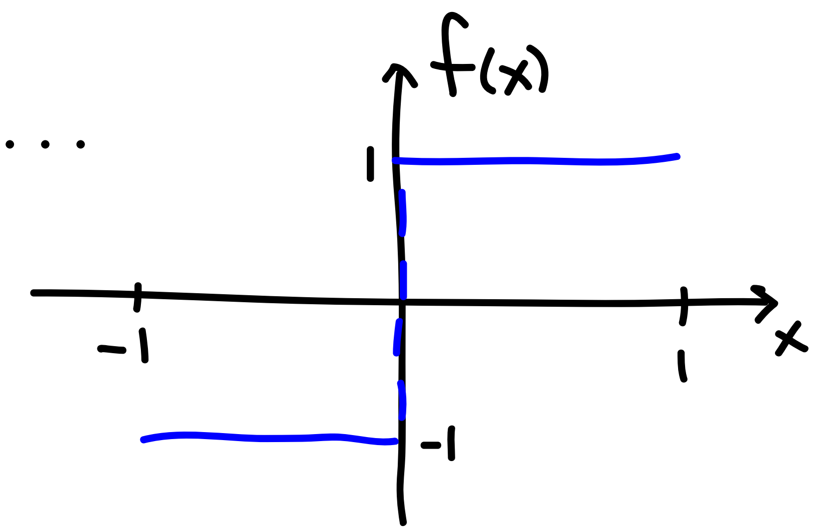

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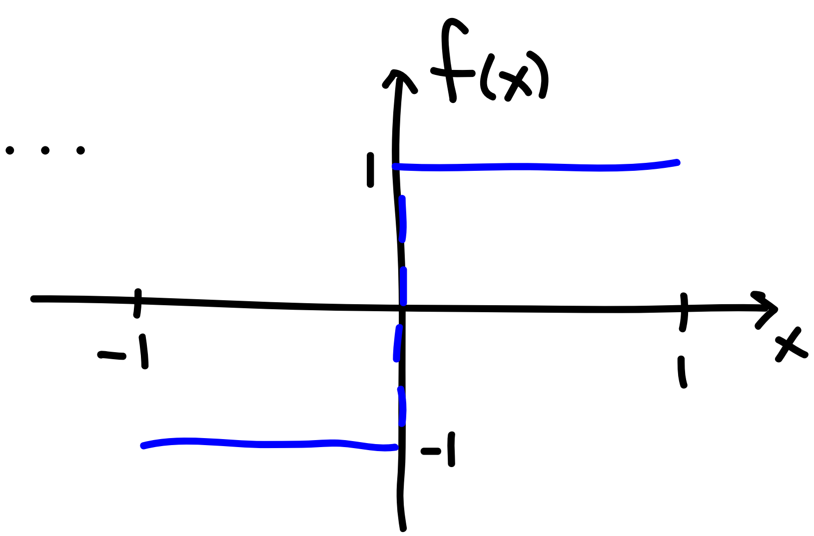

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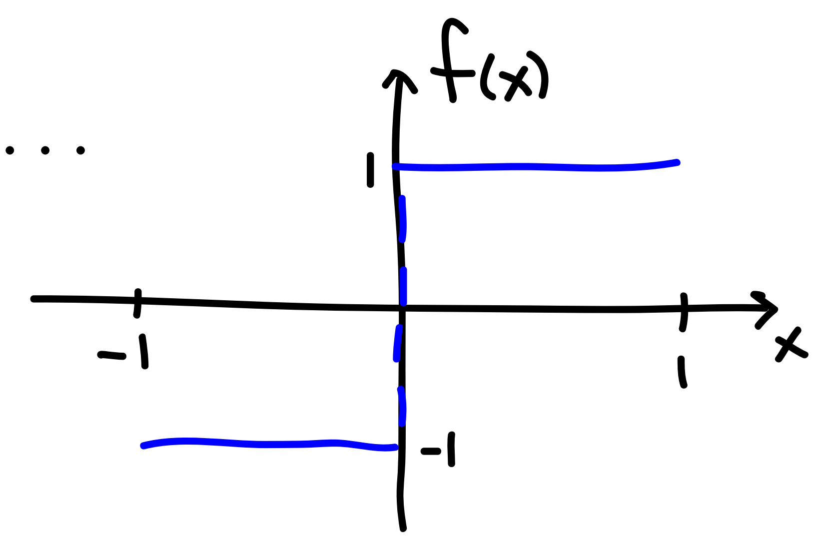

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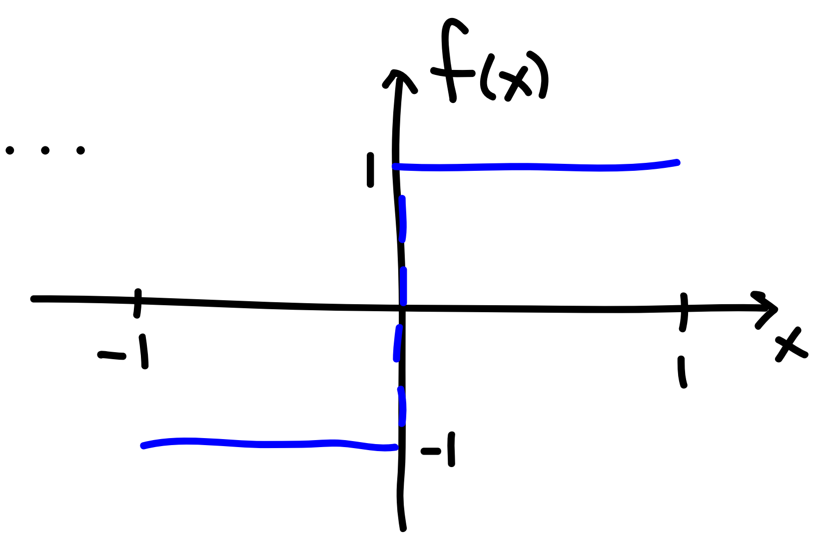

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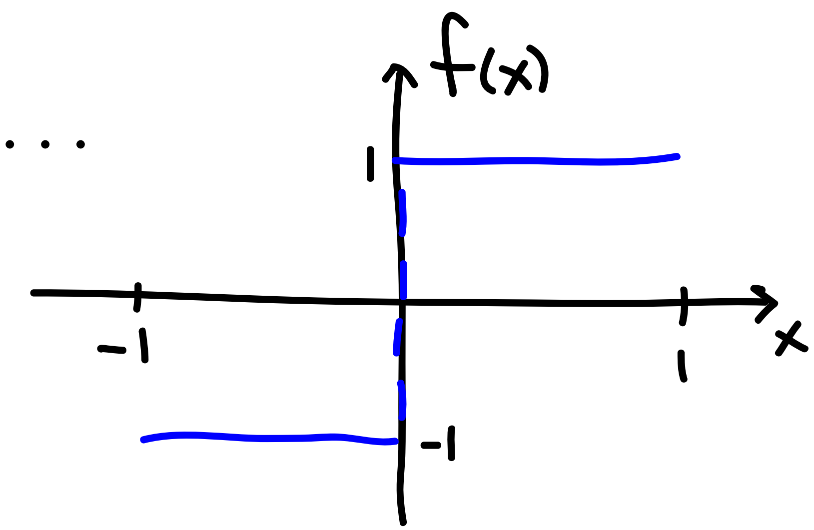

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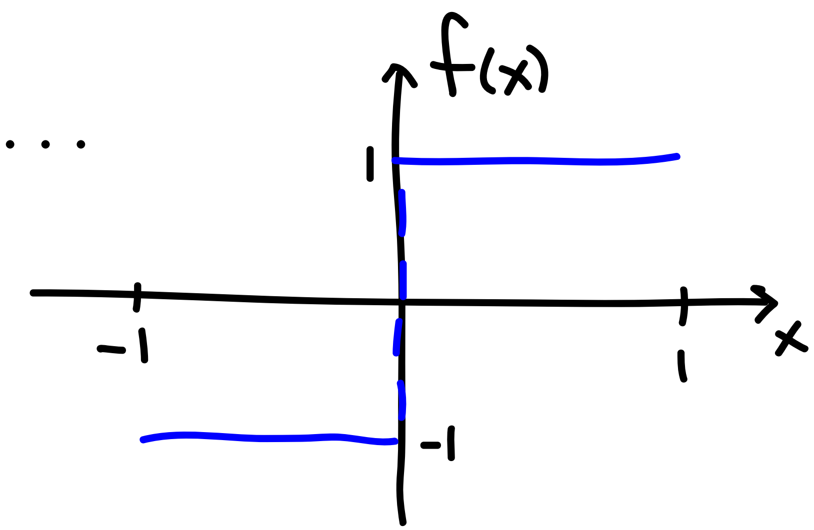

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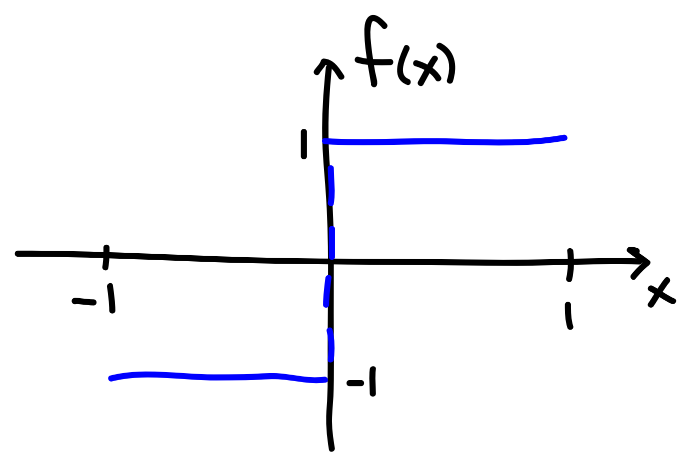

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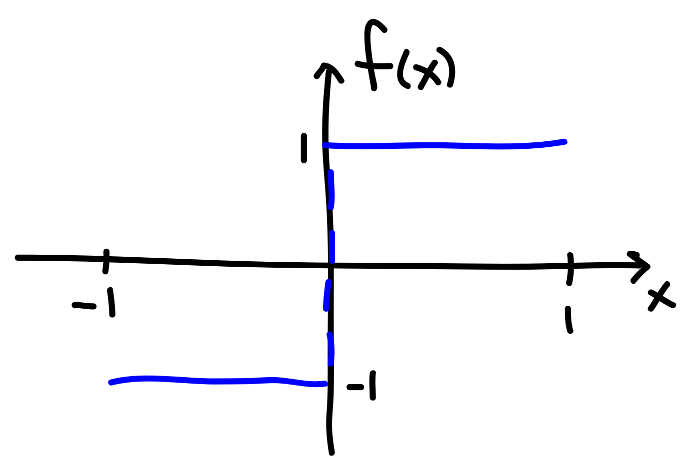

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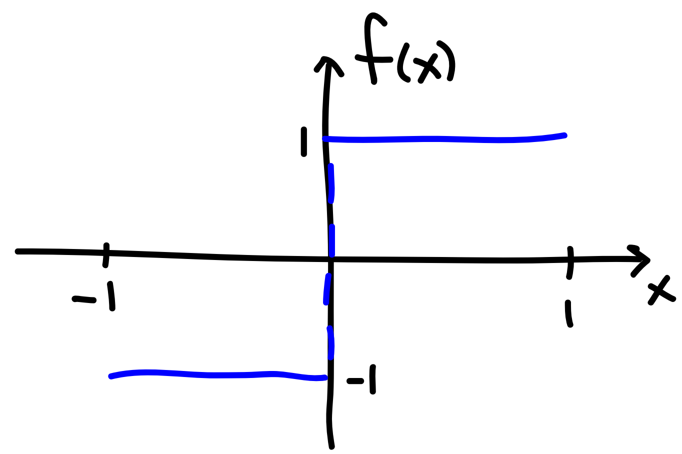

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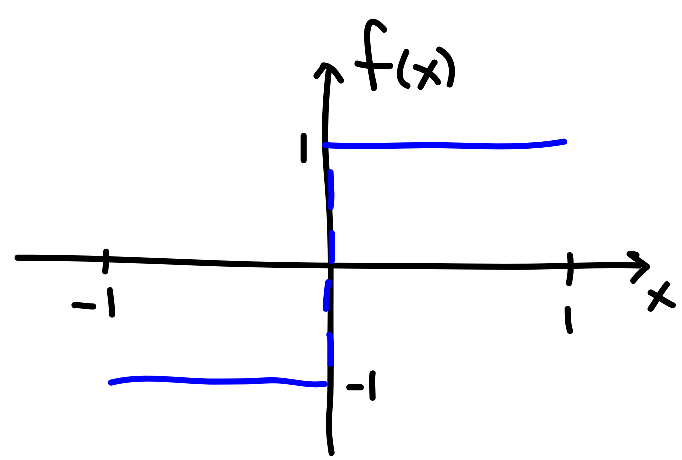

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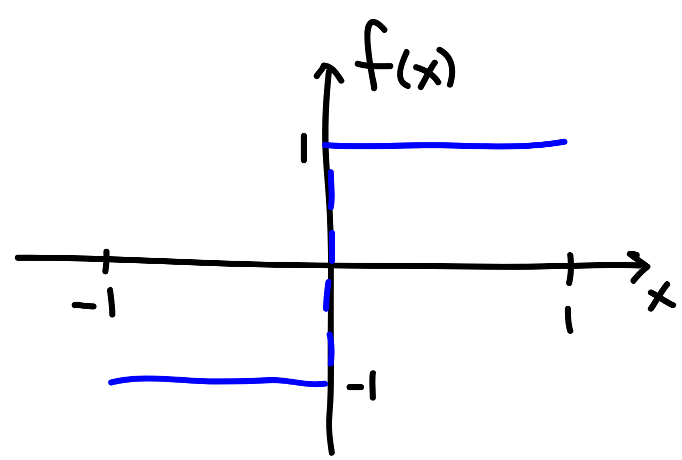

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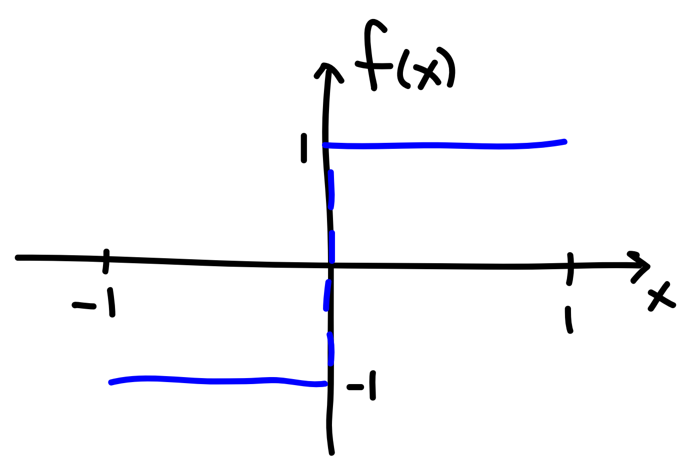

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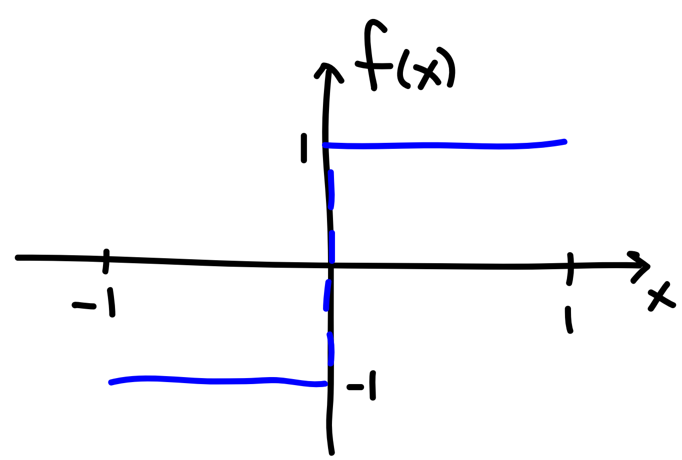

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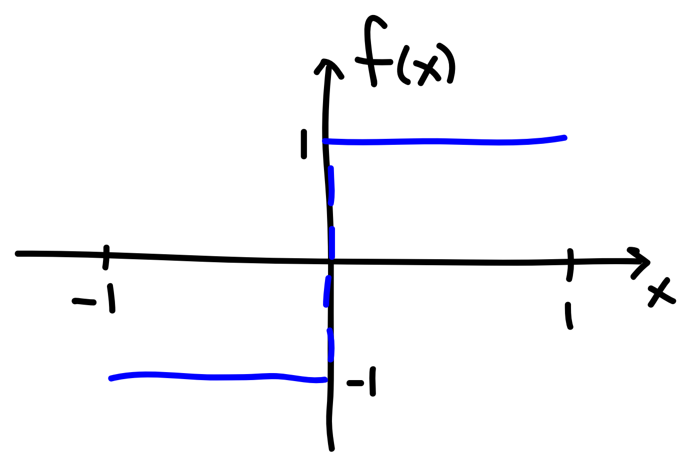

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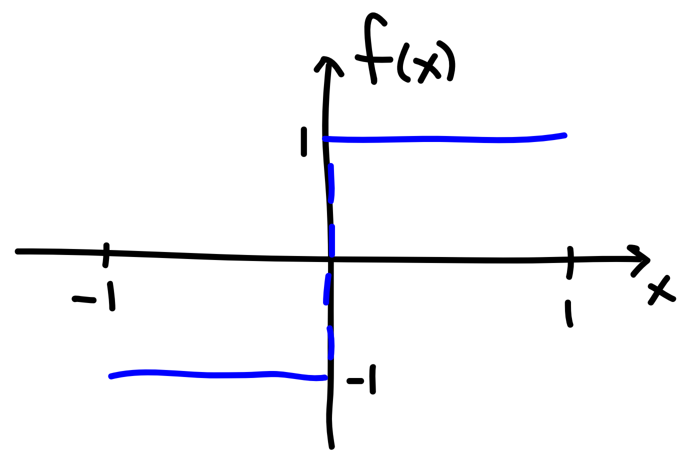

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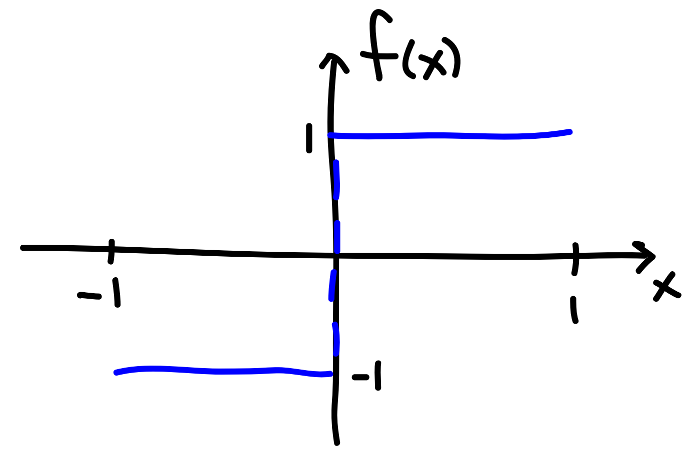

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$$b_n = \begin{cases} \dfrac{4}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

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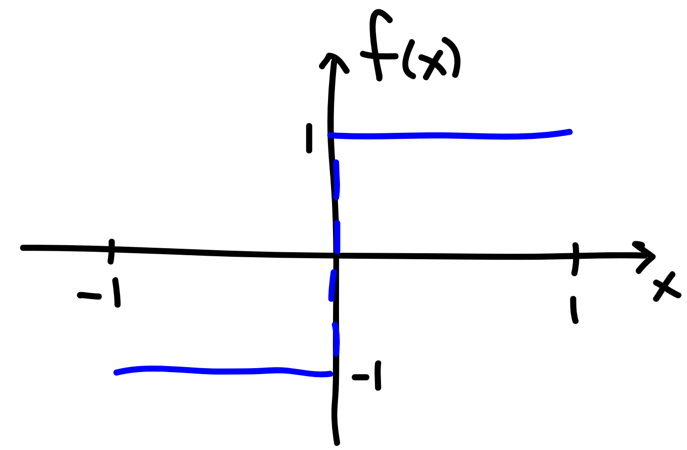

$b_n =$

(A) 0

(B) $\frac{2}{n\pi}$

(C) $\frac{4(-1)^n}{n\pi}$

☆(D) $\frac{2(1-(-1)^n)}{n\pi}$

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

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https://www.desmos.com/calculator/tlvtikmi0y

Fourier series

- Calculate the coefficients.

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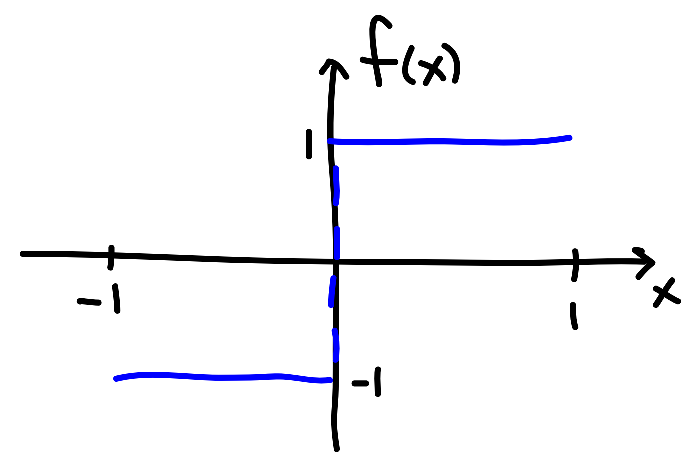

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Does f(x) = f_{FS}(x) for all x?

Fourier series

- Calculate the coefficients.

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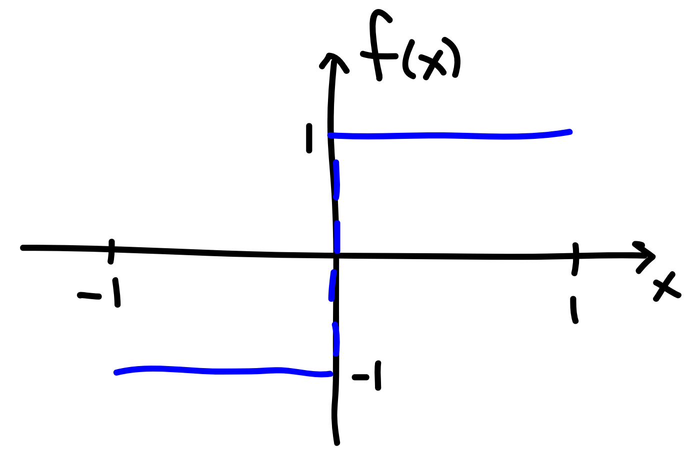

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Does f(x) = f_{FS}(x) for all x?

Problems at jumps! x=-1, 0, 1

Fourier series

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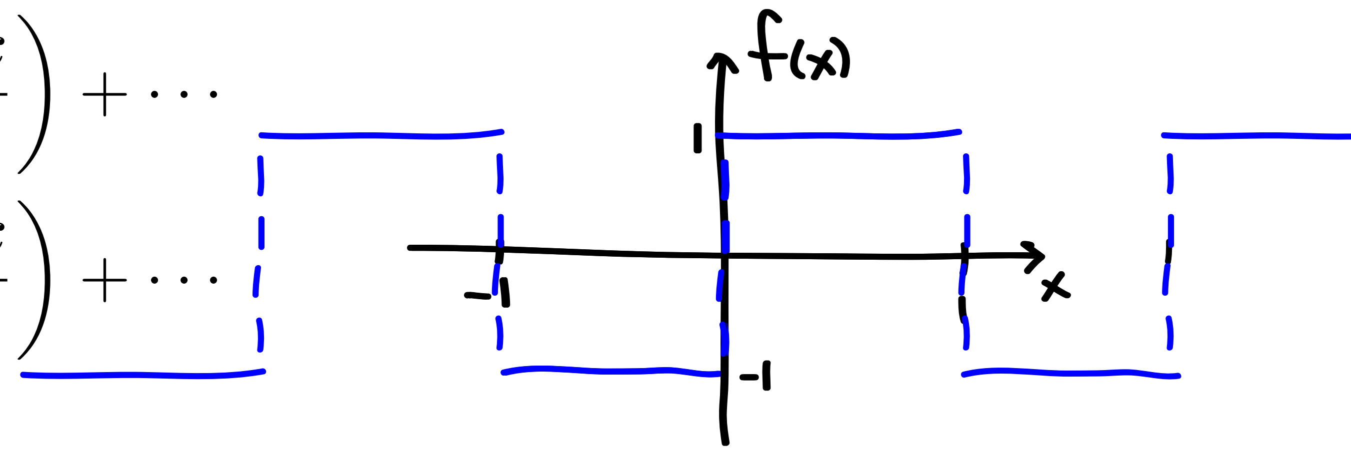

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Does f(x) = f_{FS}(x) for all x?

Problems at jumps! x=-1, 0, 1

Fourier series

- **Theorem** Suppose f anf f’ are piecewise continuous on [-L,L] and periodic beyond that interval. Then f(x) = f_{FS}(x) at all points at which f is continuous. Furthermore, at points of discontinuity, f_{FS}(x) takes the value of the midpoint of the jump. That is,

$$f_{FS}(x) = \frac{f(x^+) + f(x^-)}{2}$$

Examples