MATH 256 – Midterm 1 – February 2, 2016.

Last name: Solution/marking key First name: ______________ Student #: ______________

Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. **[3 pts]** Consider the equation $y'' + 6y' + ky = 0$. Place (a), (b) and (c) in the boxes below to correctly complete the sentences.

(a) $k > 9$

(b) $k = 9$

(c) $k < 9$

When $\boxed{c}$, the general solution is $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

When $\boxed{a}$, the general solution is $y(t) = e^{\alpha t}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right)$.

When $\boxed{b}$, the general solution is $y(t) = C_1 e^{rt} + C_2 t e^{rt}$.

2. **[5 pts]** Find the solution to the equation $y' = x/y$ subject to the initial condition $y(0) = -2$.

$y' = \frac{x}{y}$

$y y' = x$ (1)

(1) $\frac{1}{2}y^2 =$ (1) $\frac{1}{2}x^2 + C$

$y = \pm\sqrt{x^2 + 2C}$

$y(0) = \pm\sqrt{2C} = -2$

Use $-$ and $C = 2$

$$\boxed{y(t) = \overset{(1)}{-}\sqrt{x^2+4} \; (1)}$$

3. **[4 pts]** Show that $\sin(t)$ and $\cos(t)$ are independent functions but that $\sin(t)$ and $\cos(t + \frac{\pi}{2})$ are not independent. Recall that to show dependence, you must find constants C_1 and C_2 that make a linear combination of the functions zero.

$W(\sin t, \cos t) = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1 \neq 0$ (1) (1)

therefore $\sin t$ and $\cos t$ are indep.

$\sin t = $ (1) $-\cos(t + \frac{\pi}{2})$ so $C_1 \sin t + C_2 \cos(t + \frac{\pi}{2}) = 0$

for $C_1 = C_2 = 1$. (1)

Do not write in these boxes - for marking purposes only. 1: ☐ 2: ☐ 3: ☐

4. **[4 pts]** Find the general solution to the equation $tw' - w = 0$.

$w' - \frac{1}{t}w = 0$ ①

$\mu = e^{-\int \frac{1}{t} dt} = e^{-\ln t} = \frac{1}{t}$ ①

$\frac{1}{t}w' - \frac{1}{t^2}w = 0$

$\left(\frac{1}{t}w\right)' = 0$ ①

$\frac{1}{t}w = C$

$\boxed{w = Ct}$ ①

5. **[6 pts]** Use *Reduction of Order* to find a second solution to the equation $t^2y'' - 3ty' + 3y = 0$ given that $y_1(t) = t$. Along the way, you should find yourself faced with the equation $tw' - w = 0$. You may refer to your answer to the previous problem at that point.

$y_2 = vt$ ①

$y_2' = v't + v$

$y_2'' = v''t + 2v'$

$t^2(v''t + 2v') - 3t(v't + v) + 3vt = 0$

$t^3v'' - t^2v' - \cancel{3vt} + \cancel{3vt} = 0$ ①

① $tv'' - v' = 0$ Let $w = v'$.

① $tw' - w = 0$

$v' = w = ct$ (see #4 above)

① $v = c_1t^2 + c_2$

$y_2 = c_1t^3 + c_2t$ or $y_2 = t^3$ ①

Do not write in these boxes - for marking purposes only. 4: 5:

6. **[5 pts]** The equation for the motion of a mass spring system is $y'' + 3y' + 2y = f(t)$. For each $f(t)$ in the table below, give the form of the particular solution. The undetermined coefficients do not need to be calculated.

$f(t)$	$y_p(t)$
e^{2t}	Ae^{2t}
te^{4t}	$(At+B)e^{4t}$
3	A
e^{-t}	Ate^{-t}
te^{-2t}	$t(At+B)e^{-2t}$

7. **[4 pts]** Brij is developing a bioremediation process to clean up sewage spills. He sets up a trial experiment in the fountain on University Boulevard in which he pours sewage into the water at a rate of 30 litres per hour. The sewage contains bacteria at a concentration of 4 grams of bacteria per litre. Brij also adds a bacteria-eating algae to the water that can filter the water at a rate of 10 litres per hour, removing all bacteria from filtered water. The fountain initially holds 30000 litres of water. Write down a differential equation that describes the change in mass of bacteria in the fountain.

$$\frac{db}{dt} = 30\,\cancel{L}/\text{hr} \cdot 4g/\cancel{L} - \frac{10\,L/\text{hr.}\ b}{30000+30t\ L}$$

$$= 120 - \frac{b}{3000+3t} \quad (g/hr)$$

① for form of eq. $b' = (\quad) - (\quad)$

① " first term

① " "10b" in second term

① " volume in " "

Do not write in these boxes - for marking purposes only. 6: 7:

8. The differential equation $y' + f(t)y = g(t)$ has the general solution $y(t) = (C - \cos(t))/t$ where C is an arbitrary constant.

(a) **[1 pt]** What is the general solution to the equation $y' + f(t)y = 0$?

$$y_h = \frac{C}{t} \quad 1$$

(b) **[2 pts]** What is $f(t)$?

$$y_h' + f(t)y_h = 0$$

$$-\frac{C}{t^2} + f(t)\frac{C}{t} = 0 \quad 1$$

$$\boxed{f(t) = \frac{1}{t}} \quad 1$$

Other valid methods ok.

(c) **[2 pts]** What is $g(t)$?

$$y' + \frac{1}{t}y = g(t)$$

No easy points

$$ty' + y = tg(t)$$

$$(ty)' = tg(t)$$

$$ty = \int tg(t)\,dt + C \quad 1$$

$$y = \underbrace{\frac{1}{t}\int tg(t)\,dt}_{\frac{-\cos t}{t}} + \frac{C}{t}$$

$$\int tg(t)\,dt = -\cos t$$

$$tg(t) = \sin t$$

$$\boxed{g(t) = \frac{1}{t}\sin t} \quad 1$$

Other valid methods ok.

Do not write in these boxes - for marking purposes only. 8: ☐ Total: ☐ (out of 36)