Today

- General solutions, independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Homog. eq. with constant coeff. (Section 3.1)

- Which of the following functions are also solutions?

  (A) $y(t) = y_1(t)^2$

  ☆ (B) $y(t) = y_1(t)+y_2(t)$

  (C) $y(t) = y_1(t)\, y_2(t)$

  (D) $y(t) = y_1(t) / y_2(t)$

  Last class, we found that if $y_1(t)$ is a solution to

  $$ay'' + by' + cy = 0$$

  then so is $y(t) = C_1y_1(t)$.

- In fact, the following are all solutions: $C_1y_1(t)$, $C_2y_2(t)$, $C_1y_1(t)+C_2y_2(t)$.
- With first order equations, the arbitrary constant appeared through an integration step in our methods. With second order equations, not so lucky.
- Instead, find two independent solutions, $y_1(t)$, $y_2(t)$, by whatever method.
- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.

Homog. eq. with constant coeff. (Section 3.1)

- One case where the arbitrary constants DO appear as we calculate:

$$y'' + y' = 0$$

$$y' + y = C_1$$

$$e^t y' + e^t y = C_1 e^t$$

$$(e^t y)' = C_1 e^t$$

$$e^t y = C_1 e^t + C_2$$

$$y = C_1 + C_2 e^{-t}$$

- More common would be that we find solutions y(t) = 1 and y(t)= e^{-t} and simply write down

$$y = C_1 + C_2 e^{-t}$$

Homog. eq. with constant coeff. (Section 3.1)

- So in general how do we find the two independent solutions y_1 and y_2?
- Exponential solutions seem to be common so let's assume $y(t)=e^{rt}$ and see if that gets us anything useful..
- Solve $y'' + y' = 0$ by assuming $y(t) = e^{rt}$ for some constant r.

$$(e^{rt})'' + (e^{rt})' = 0$$
$$r^2 e^{rt} + r e^{rt} = 0$$
$$r^2 + r = 0$$
$$r(r+1) = 0$$
$$r = 0, \quad r = -1$$

$$y = C_1 e^0 + C_2 e^{-t}$$
$$y = C_1 + C_2 e^{-t}$$

Homog. eq. with constant coeff. (Section 3.1)

- Solve $y'' - 4y = 0$ subject to the ICs $y(0) = 3, y'(0) = 2$.

(A) $y(t) = C_1 e^{2t} + C_2 e^{-2t}$

(B) $y(t) = 2e^{2t} + e^{-2t}$

(C) $y(t) = \dfrac{7}{4} e^{4t} + \dfrac{5}{4} e^{-4t}$

(D) $y(t) = e^{2t} + 2e^{-2t}$

(E) $y(t) = C_1 e^{4t} + C_2 e^{-4t}$

Homog. eq. with constant coeff. (Section 3.1)

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  i. Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)

  ii. A repeated real root: $b^2 - 4ac = 0$.

  iii. Two complex roots: $b^2 - 4ac < 0$.

- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0, \; y'(0) = v_0$ with these two solutions?

$$y(t) = C_1 e^{2t+3} + C_2 e^{2t-3}$$

$$y(0) = C_1 e^3 + C_2 e^{-3} = y_0$$

$$y'(0) = 2C_1 e^3 + 2C_2 e^{-3} = v_0$$

- Solve this system for C_1, C_2...

- Can't do it. Why? $\begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$

$$\det \begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} = 0$$

Independence and the Wronskian (Section 3.2)

- For any two solutions to some linear ODE, to ensure that we have a general solution, we need to check that

$$\det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$$

- For ICs other than t_0=0, we require that

$$y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$$

- This quantity is called the Wronskian.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

Independence and the Wronskian (Section 3.2)

- Two functions $y_1(t)$ and $y_2(t)$ are linearly independent provided that the only way that $C_1y_1(t) + C_2y_2(t) = 0$ for all values of t is when $C_1=C_2=0$.

  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

  Find values of $C_1\neq 0$ and $C_2\neq 0$ so that $C_1y_1(t) + C_2y_2(t) = 0$.

(A) $C_1 = e^{-2t-3},\ C_2 = -e^{-2t+3}$

(B) $C_1 = e^{-2t+3},\ C_2 = -e^{-2t-3}$

(C) $C_1 = e^{-3},\ C_2 = e^{3}$

☆ (D) $C_1 = e^{-3},\ C_2 = -e^{3}$

(E) $C_1 = e^{3},\ C_2 = -e^{-3}$

Independence and the Wronskian (Section 3.2)

- Two functions $y_1(t)$ and $y_2(t)$ are linearly independent provided that the only way that $C_1y_1(t) + C_2y_2(t) = 0$ for all values of t is when $C_1=C_2=0$.

  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

- The Wronskian is defined for any two functions, even if they aren't solutions to an ODE.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

- If the Wronskian is nonzero for some t, the functions are linearly independent.

- If $y_1(t)$ and $y_2(t)$ are solutions to an ODE and the Wronskian is nonzero then they are independent and

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions and we can use them to solve any IC.

Independence and the Wronskian (Section 3.2)

- So for case i (distinct roots), can we form a general solution from

$$y_1(t) = e^{r_1 t} \text{ and } y_2(t) = e^{r_2 t}\text{?}$$

- Must check the Wronskian:

$$W(e^{r_1 t}, e^{r_2 t})(t) = e^{r_1 t} r_2 e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$$= (r_1 - r_2) e^{r_1 t} e^{r_2 t} \neq 0$$

So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.

Independence and the Wronskian (Section 3.2)

- Example: Consider the equation $y'' + 9y = 0$. Find the roots of the characteristic equation (i.e. the r values).

(A) $r_1 = 3$, $r_2 = -3$.

(B) $r_1 = 3$ (repeated root).

★ (C) $r_1 = 3i$, $r_2 = -3i$.

(D) $r_1 = 9$, (repeated root).

As we'll see soon, this means that $y_1(t) = \cos(3t)$ and $y_2(t)=\sin(3t)$.

Do these form a fundamental set of solutions? Calculate the Wronskian.

$$W(\cos(3t), \sin(3t))(t) =$$

(A) 0 ★ (C) 3

(B) 1 (D) $2\cos(3t)\sin(3t)$

Distinct roots - asymptotic behaviour (Section 3.1)

- Three cases:

(i) Both r values positive.

e.g. $y(t) = C_1 e^{2t} + C_2 e^{5t}$

(ii) Both r values negative.

e.g. $y(t) = C_1 e^{-2t} + C_2 e^{-5t}$

(iii) The r values have opposite sign.

e.g. $y(t) = C_1 e^{-2t} + C_2 e^{5t}$

Except for the zero solution y(t)=0, the limit $\lim_{t \to \infty} y(t)$ ...

(A) ...is unbounded for all ICs.

(B) ...is unbounded for most ICs but not for a few carefully chosen ones.

(C) ...goes to zero for all ICs.

Challenge: come up with an initial condition for (iii) that has a bounded solution.

Distinct roots - asymptotic behaviour (Section 3.1)

- Three cases:

(i) Both r values positive.

e.g. $y(t) = C_1 e^{2t} + C_2 e^{5t}$

(ii) Both r values negative.

e.g. $y(t) = C_1 e^{-2t} + C_2 e^{-5t}$

(iii) The r values have opposite sign.

e.g. $y(t) = C_1 e^{-2t} + C_2 e^{5t}$

Except for the zero solution y(t)=0, the limit $\lim_{t \to -\infty} y(t)$...

★ (A) ...is unbounded for all ICs.

★ (B) ...is unbounded for most ICs but not for a few carefully chosen ones.

★ (C) ...goes to zero for all ICs.

Complex roots (Section 3.3)

- Complex number review (Euler's formula)
- Complex roots of the characteristic equation
- From complex solutions to real solutions

Complex number review

- We define a new number: $i = \sqrt{-1}$

- Before, we would get stuck solving any equation that required square-rooting a negative number. No longer.

- e.g. The solutions to $x^2 - 4x + 5 = 0$ are $x = 2 + i$ and $x = 2 - i$

- For any equation, $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$, the solutions have the form $x = \alpha \pm \beta i$ where α and β are both real numbers.

- For $\alpha + \beta i$, we call α the real part and β the imaginary part.

Complex number review

- Adding two complex numbers:

$$(a+bi)+(c+di) = \underbrace{a+c}+\underbrace{(b+d)}i$$

- Multiplying two complex numbers:

$$(a+bi)(c+di) = \underbrace{ac-bd}+\underbrace{(ad+bc)}i$$

- Dividing by a complex number:

$$(a+bi)/(c+di) = (a+bi)\frac{1}{(c+di)}$$

    - What is the inverse of c+di?

Complex number review

- What is the inverse of c+di written in the usual complex form p+qi?

(A) $c - di$

(B) $\dfrac{c+di}{c^2+d^2}$

★(C) $\dfrac{c-di}{c^2+d^2}$

(D) $\dfrac{1}{c-di}$

$$(c+di)\frac{c-di}{c^2+d^2} = \frac{c^2+d^2-(cd-dc)i}{c^2+d^2} = 1$$

- Dividing by a complex number:

$$(a+bi)/(c+di) = (a+bi)\frac{c-di}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

Complex number review

- Definitions:
  - Conjugate - the conjugate of $a + bi$ is

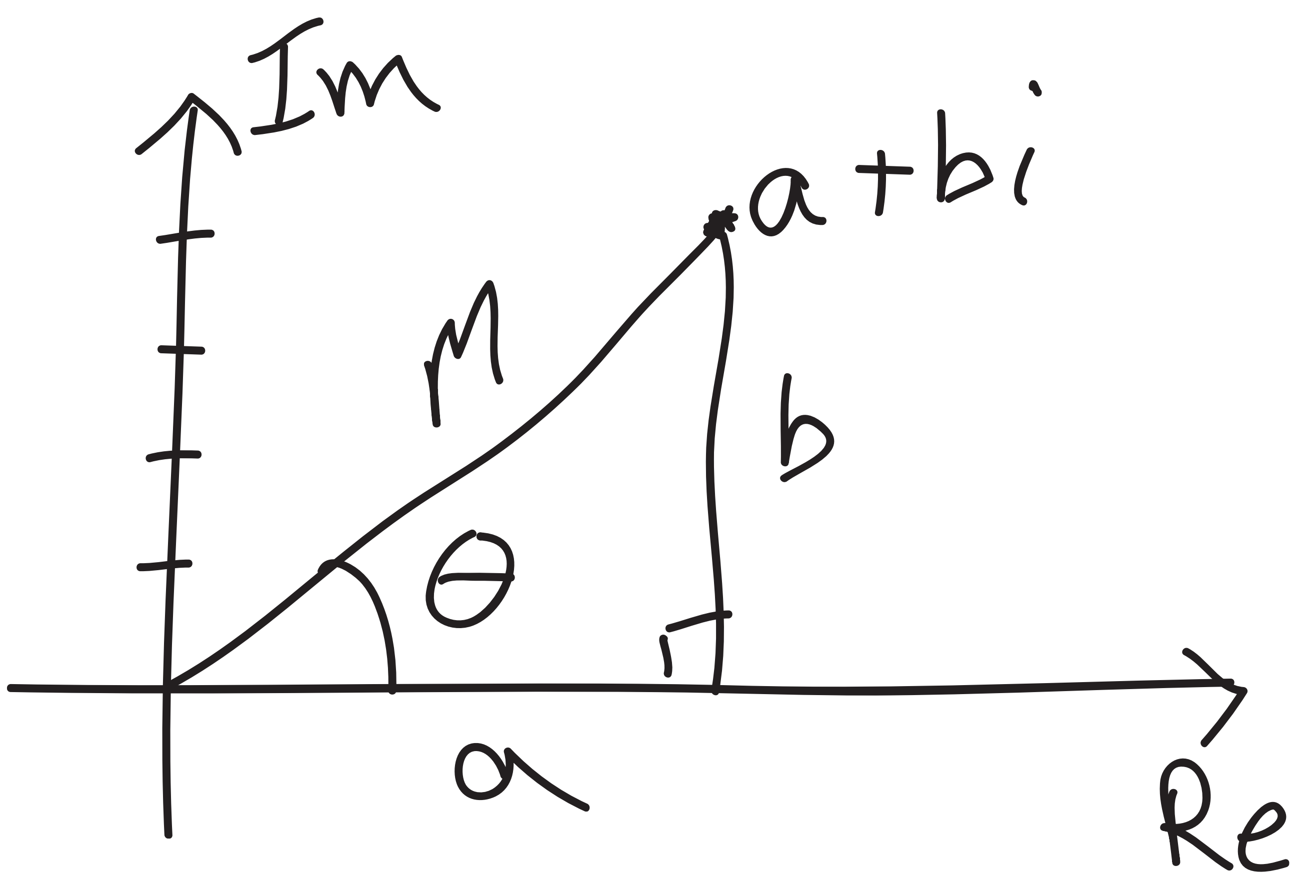
$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M\cos\theta$$

$$b = M\sin\theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a + bi = M(\cos\theta + i\sin\theta)$$

θ is sometimes called the argument or phase of $a + bi$.

Complex number review

- Toward Euler's formula
  - Taylor series - recall that a function can be represented as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

  - What function has Taylor series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(A) cos x

(C) e^x

(B) sin x

(D) ln x

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{-1 = i^2}$$

$$= 1 + i\theta + (-1)\frac{\theta^2}{2!} + (-1)i\frac{\theta^3}{3!} + (-1)^2\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

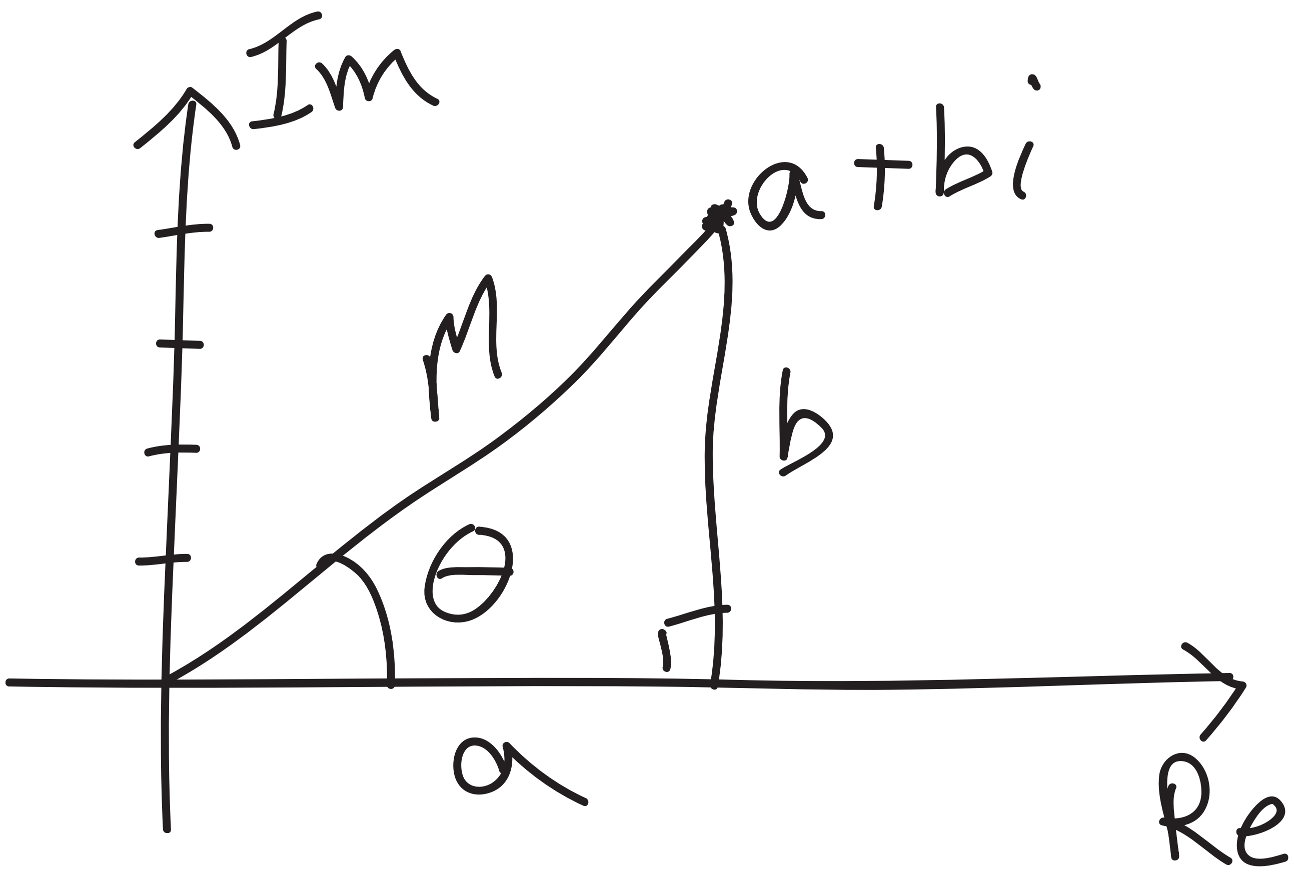
$$\cos\theta + i\sin\theta$$

Euler's formula:

$$= e^{i\theta}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M\cos\theta$$

$$b = M\sin\theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a + bi = M(\cos\theta + i\sin\theta)$$

$$a + bi = Me^{i\theta}$$

(Polar form makes multiplication much cleaner)