# Today

- Neumann BC example
- Summary of steps for solving the Diffusion Equation with homogeneous Dirichlet or Neumann BCs using Fourier Series.
- Nonhomogeneous BCs
- Eigenvalue / eigenvector interpretation
- End-of-term info:
    - Don't forget to complete the online teaching evaluation survey.
    - Review and office hours during exams TBA - by online poll

# The Diffusion equation

Solve the equation $\dfrac{dc}{dt} = D\dfrac{d^2c}{dx^2}$

subject to boundary conditions $\dfrac{\partial c}{\partial x}(0,t) = 0, \;\; \dfrac{\partial c}{\partial x}(2,t) = 0$

and initial condition $c(x,0) = x$ defined on [0,2].

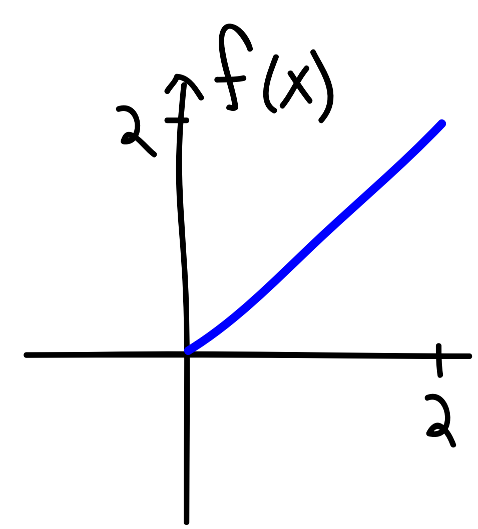
What is the steady state in this case? $c_{ss}(x) = Ax+B$

BC says A=0. B=?

$$\text{Total initial mass} = \int_0^L c(x,0)\;dx$$

$$\text{Total ``final'' mass} = \int_0^L c_{ss}(x)\;dx$$

} No flux BCs so these must be equal.




In this case, the Fourier series also tells us the answer:

$$c(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}Dt}\cos\left(\frac{n\pi x}{2}\right) \longrightarrow$$ $c_{ss}(x)=a_0/2$

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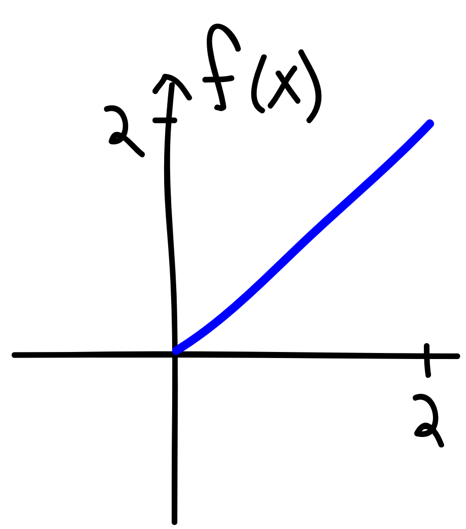

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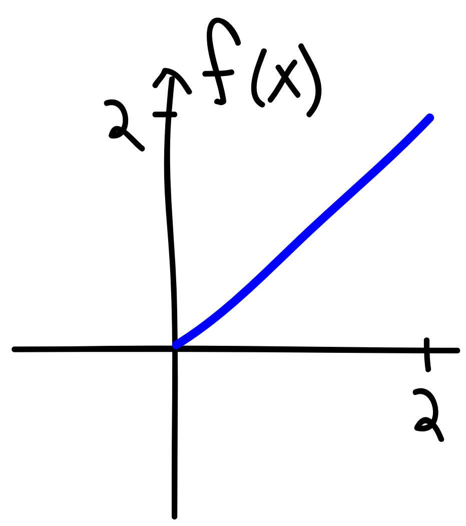

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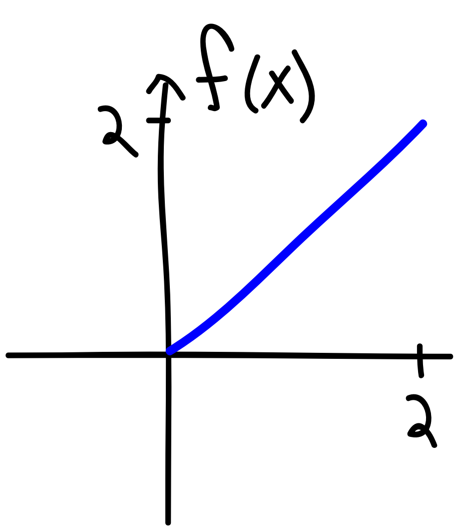

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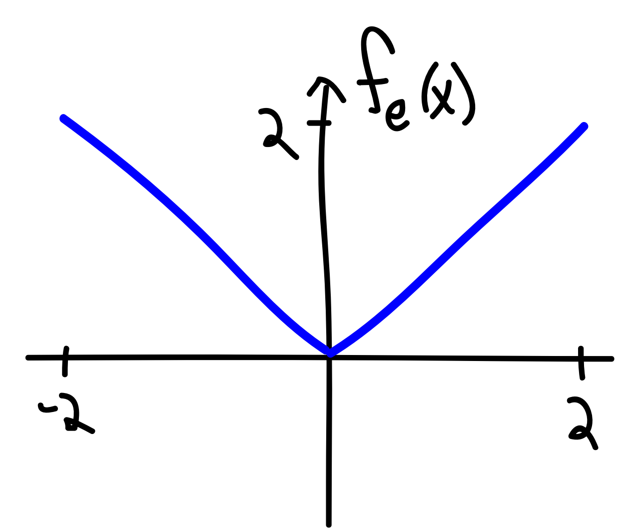

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$$a_0 = 2$$

$$a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n^2}\cos\frac{n\pi x}{2}$$

$$c(x,t) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n^2}e^{-\frac{n^2\pi^2}{4}Dt}\cos\frac{n\pi x}{2}$$

# Solving the Diffusion equation using FS - summary

- The Diffusion equation ties the time-exponent to the space-frequency:

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

$$c(x,t) = be^{-w^2Dt}\sin(wx)$$

$$d(x,t) = ae^{-w^2Dt}\cos(wx)$$

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- Boundary conditions whether you need a Fourier sine or cosine series and determines the frequency ω.

$$c(0,t) = 0,\ \ c(L,t) = 0 \ \Rightarrow c_n(x,t) = b_n e^{-\frac{n^2\pi^2}{L^2}Dt}\sin\left(\frac{n\pi x}{L}\right)$$

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- The initial condition determines the $a_n$ or $b_n$ values via Fourier series.

$$c(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$ or $$c(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

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**Solution:**

$$c(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}Dt}\cos\left(\frac{n\pi x}{L}\right)$$

- Bound... series and d...

$$c(0,t) = 0,\ c(L,t) = 0 \Rightarrow c_n(x,t) = b_n e^{\ldots} \sin\left(\frac{\ldots}{L}\right)$$

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# Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = 0$$

$$u(x,0) = \sin\frac{3\pi x}{2}$$

Write down the solution to this IVP.

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(A) $u(x,t) = e^{-9\pi^2 t}\cos\frac{3\pi x}{2}$

(B) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2}$

(C) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$ $\qquad b_n = \int_0^2 \sin\frac{3\pi x}{2}\sin\frac{n\pi x}{2}\,dx$

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(C) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$ ← don't satisfy BCs.

$$b_n = \int_0^2 \sin\frac{3\pi x}{2}\sin\frac{n\pi x}{2}\,dx$$

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# ...with nonhomogeneous boundary conditions

$$u_t = Du_{xx}$$

$$u(0,t) = 0$$
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$$u(x,0) = x^2$$

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Still use sin(nπx/L) but need to get end(s) away from zero!

What is steady state?

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Ultimately, we want $u(x,t) = 2x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin \frac{n\pi x}{L}$

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What function do we use to calculate the Fourier series $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ ?

(A) $x^2$ (B) $x^2 - 2$ (C) $x^2 - 2x$ (D) $x^2 + 2x$

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- Solve the Diffusion Equation with nonhomogeneous BCs:

$$u_t = D u_{xx}$$
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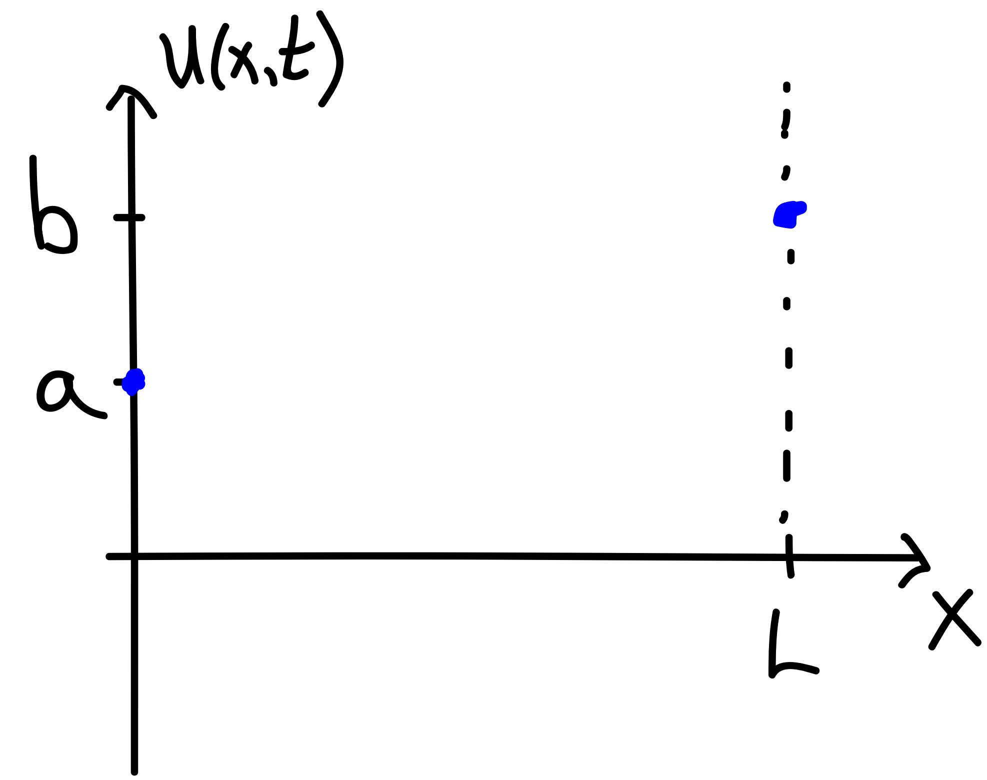

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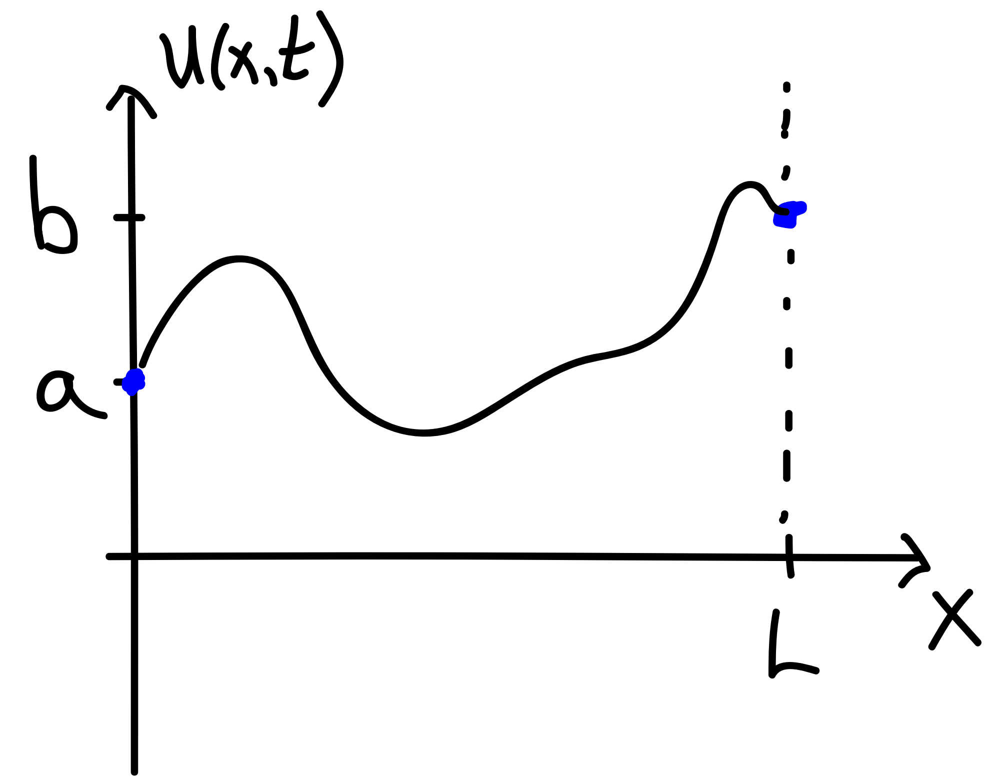

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- Solve the Diffusion Equation with nonhomogeneous BCs:

$$u_t = Du_{xx}$$
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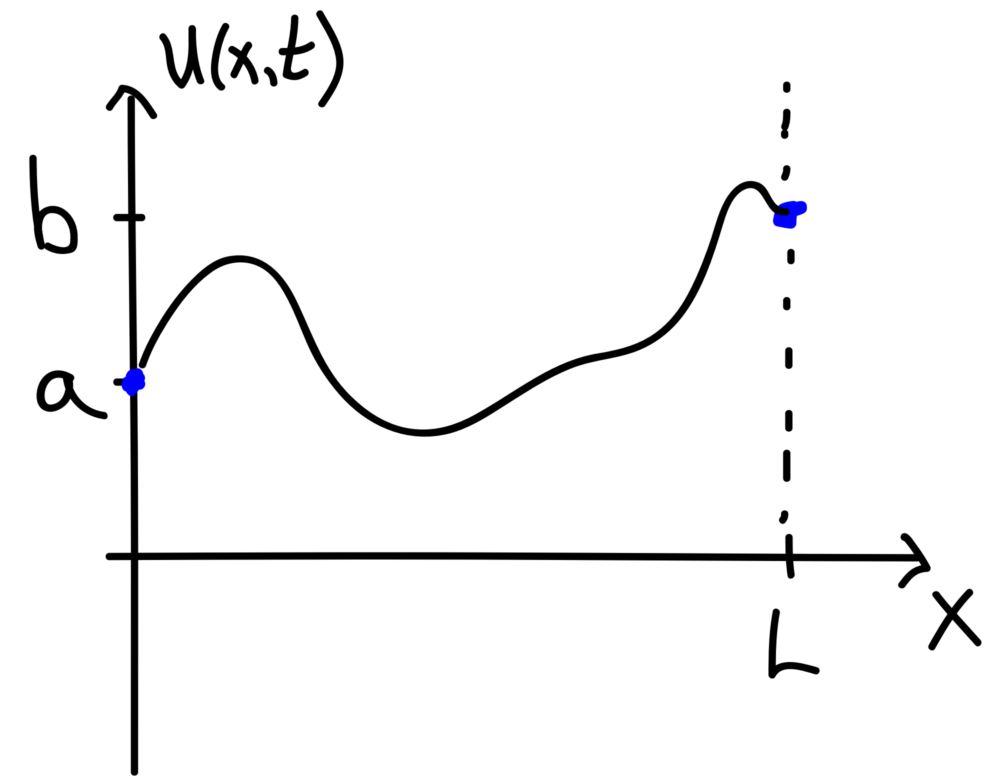
- Recall - rate of change is proportional to concavity so bumps get ironed out.



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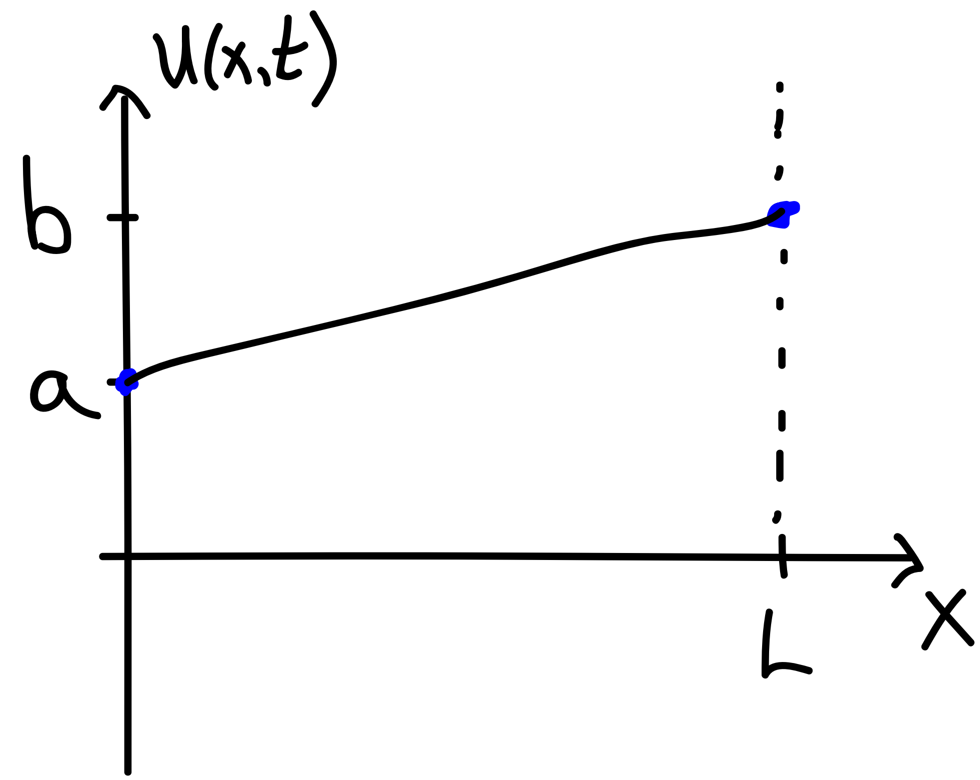

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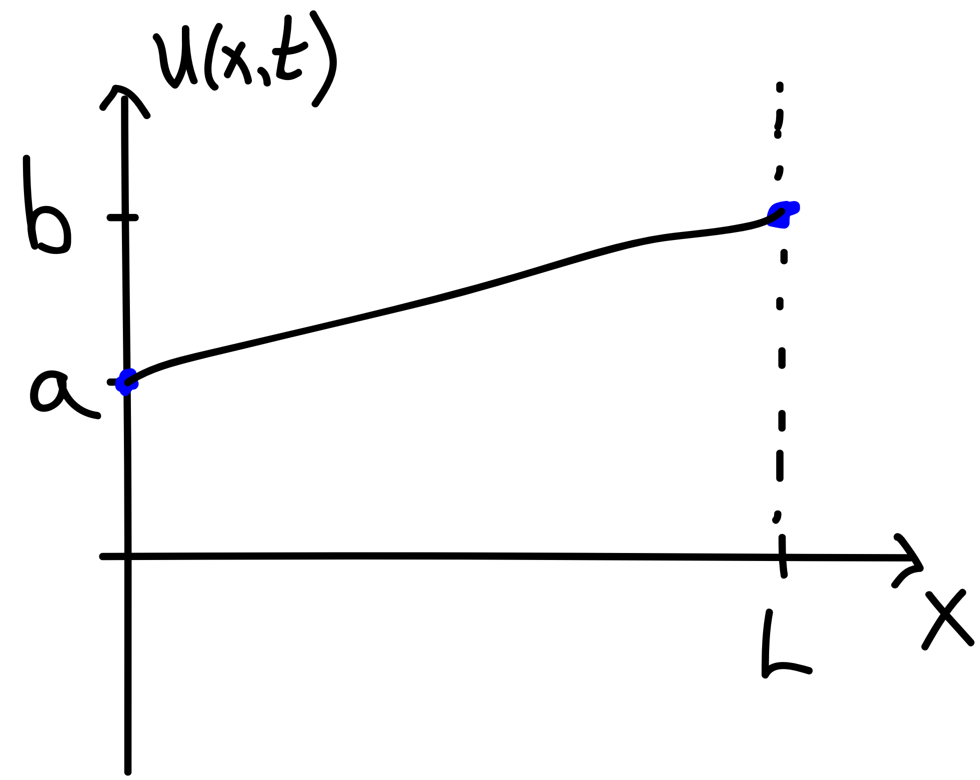


$$v(x,t) = u(x,t) - \left(a + \frac{b-a}{L}x\right)$$

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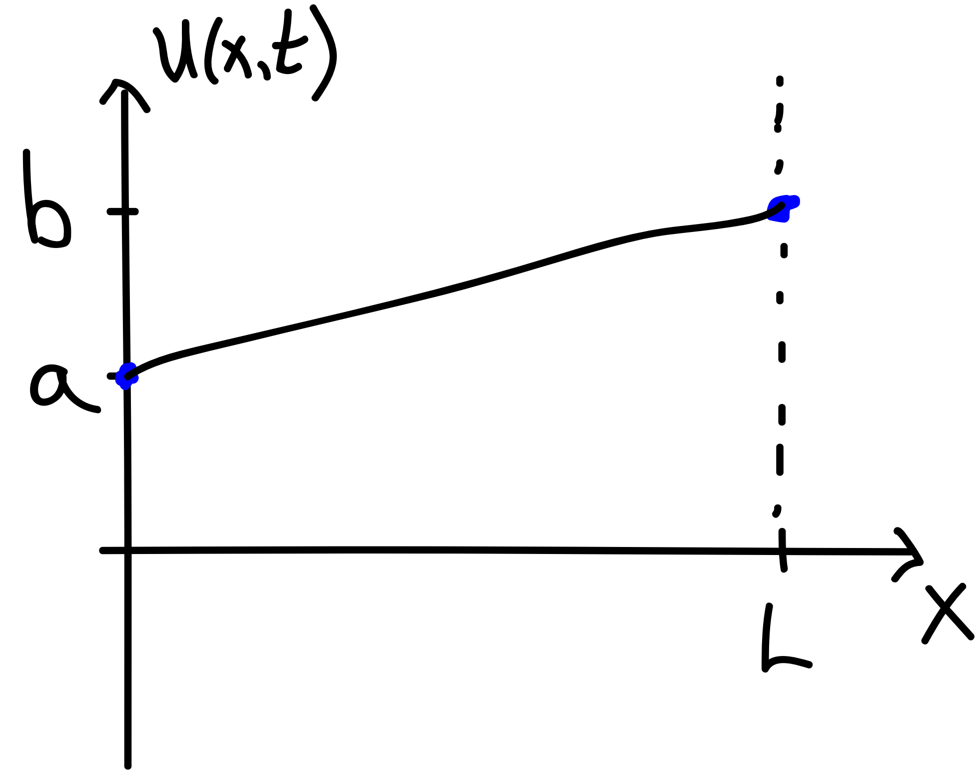


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$$\left.\begin{aligned} v_t &= u_t \\ v_{xx} &= u_{xx} \end{aligned}\right\} \Rightarrow v_t = D v_{xx}$$

$$v(0,t) = u(0,t) - a = 0$$
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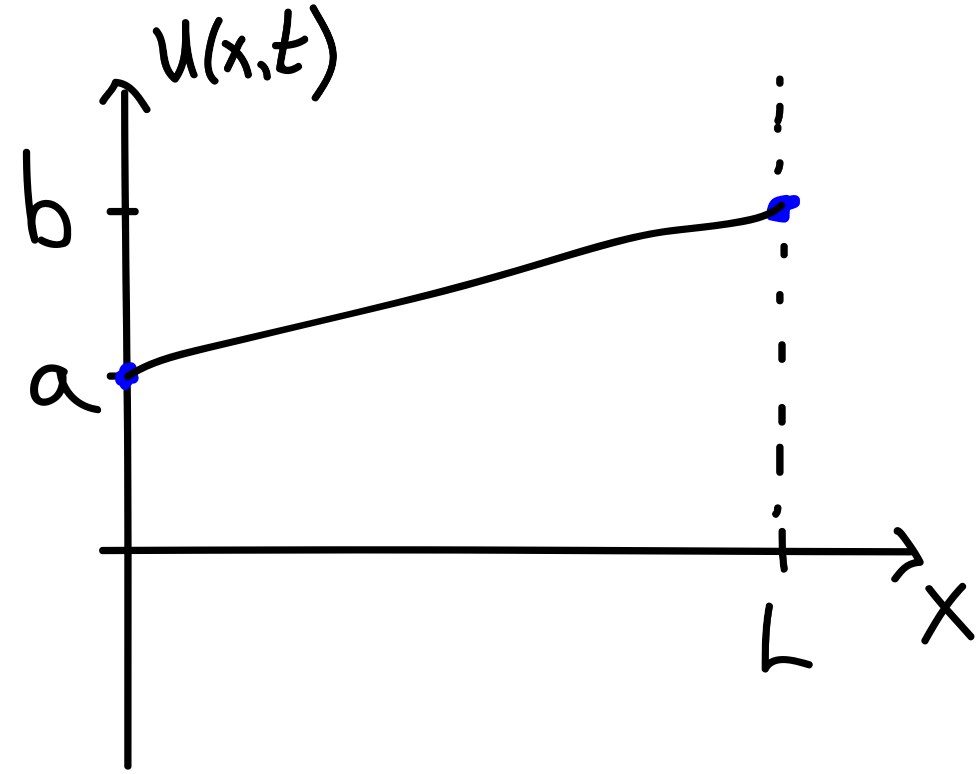


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- v(x,t) satisfies the Diffusion Eq with homogeneous Dirichlet BCs and a new IC.

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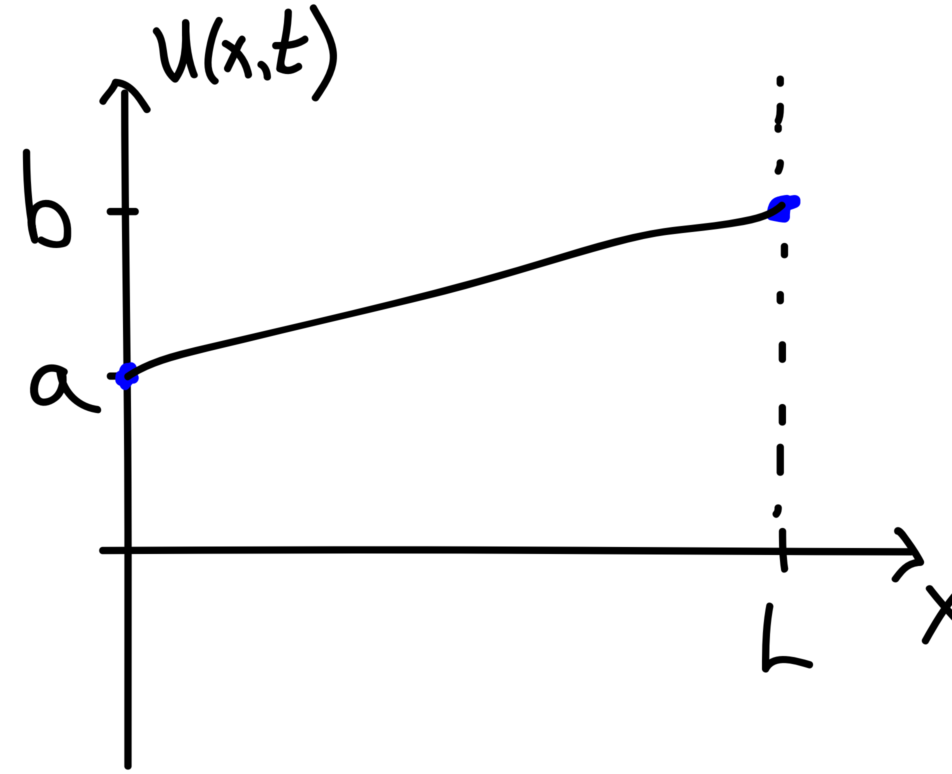


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- General trick: define v=u-SS and find v as before.

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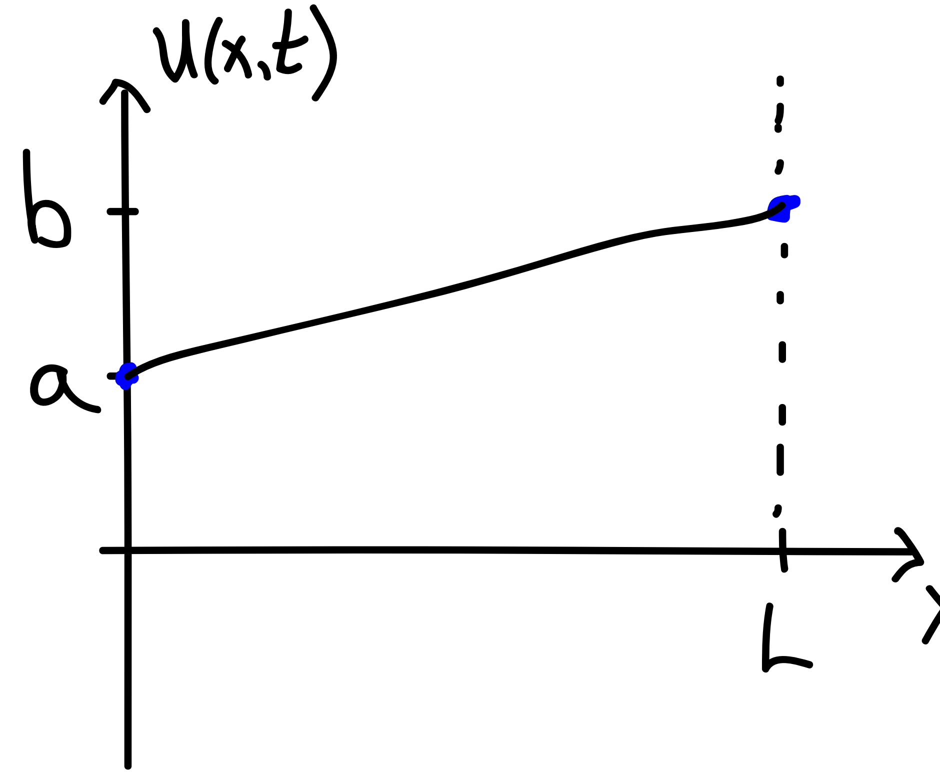


$$v(x,t) = u(x,t) - \left(a + \frac{b-a}{L}x\right)$$

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https://www.desmos.com/calculator/6jp7jggsf9