Quantum K-theory of git quotients

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Chris Woodward, Rutgers University New Brunswick, joint with Eduardo Gonzalez, U Mass Boston

Equivariant cohomology

G connected complex reductive group with Lie algebra $\mathfrak{g}$

V smooth (quasi)-projective G-variety

$EG \to BG$ universal G bundle, $V_G = EG \times_G V$ classifying space

$H_G(V) = H(V_G)$ (Say rational coefficients.)

If G action is free then $H(V_G) = H(V/G)$ (spectral sequence)

Cohomology of BG

$H(BG) = Sym(\mathfrak{g}^\vee)^G$ invariant polynomials

Example: G torus, $BG \cong (\mathbb{C}P^\infty)^k, H_G(V) = Sym(\mathfrak{g}^\vee)$.

$G = \mathbb{C}^\times, V = \mathbb{C}^k, H_G(V) = \mathbb{Q}[h]$.

GIT QUOTIENTS

V smooth (quasi)-projective G-variety

$\mathcal{O}_V(1)$ ample G-line bundle

V^{ss} points with non-vanishing invariant sections of $\mathcal{O}_V(1)^{\otimes k}, k > 0$

$V /\!\!/ G = V^{\mathrm{ss}}/ \sim$ orbit equivalence relation

$= V^{\mathrm{ss}}/G$ if action has finite stabilizers

EXAMPLE: TORIC VARIETIES

Example: G torus

V finite diml representation of G with weights in an open half space in $\mathfrak{g}^\vee$

$\mathrm{Pic}_G(V) \cong \mathfrak{g}^\vee_{\mathbb{Z}} \in \omega$ class of $\mathcal{O}_V(1)$

$x \in V^{\mathrm{ss}}$ semistable iff the span of weights corresponding to $x_i \neq 0$ contains ω.

Any toric variety arises this way

Example: projective space

$V = \mathbb{C}^{k+1}, G = \mathbb{C}^\times$ all weights one.

$\omega = 1 \in \mathfrak{g}_{\mathbb{R}}^\vee$.

Every point except $0 \in V$ is semistable.

$V /\!\!/ G = \mathbb{C}^{k+1} - \{0\} = \mathbb{P}^k$.

Example

Grassmannians

$X = \mathrm{Gr}(k, n)$ the Grassmannian of k-planes in $\mathbb{C}^n$.

Then $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) /\!\!/ SL(k) = V /\!\!/ G$

The semistable locus for suitable $\mathcal{O}_v(1)$ consists of maps of maximal rank.

Kirwan's method for obtaining presentations

$H_G(V) := H(V_G)$ restricts to $H_G(V^{\mathrm{ss}}) \cong H(V/G)$.

Kirwan: $H_G(V) \to H(V /\!\!/ G)$ surjective

Corollary: If V is a vector space then $H(V /\!\!/ G)$ is a quotient of $H(BG) = Sym(\mathfrak{g}^\vee)^G$.

Sketch of proof

V has a stratification by "direction of maximal instability"

This stratification turns out to be "equivariantly perfect" so $H_G(V)$ is a sum of contributions from strata, one of which is $H(V /\!\!/ G)$.

Example: $V = \mathbb{C}^{k+1}$ then $H_G(V) = H(\mathbb{P}^k) \oplus h^{k+1} H_G(\{0\})$

so $H(\mathbb{P}^k) = \mathbb{Q}[h]/h^{k+1}$

Cohomology of toric varieties

X smooth polarized projective toric variety: variety with the action of a torus G with an open orbit.

naturally maps via "moment map" to P convex hull of weights of G on $H^0(\mathcal{O}_X(1))$

$F_1, \ldots, F_k$ facets of P

preimages $D_1, \ldots, D_k$ boundary divisors

Danilov, Jurkiewicz: $H(X)$ generated by $H^2(X)$ modulo $\prod_{i\in I}[D_i]$ whenever $\cap_{i\in I} F_i$ is empty

Proof: Generation by Kirwan. Relations: Each vertex of P corresponds to a cell whose closure is a cohomology generator and is the inverse image of a face F, which is the intersection of facets $F_1, \ldots, F_j$

$[\Phi^{-1}(F)] = [\Phi^{-1}(F_1) \cap \ldots \cap \Phi^{-1}(F_j)] = D_1 \ldots D_j$. In particular, if the intersection is empty then $D_1 \ldots D_j = 0$.

Showing these are all relations requires an induction which is actually easier in the quantum setting to come.

K-THEORY KIRWAN MAP

$K_G(V)$ equivariant vector bundles (or coherent sheaves) on V, rational coefficients (topological version related to $K(V_G)$ by formal completion)

$K_G(V) \to K(V/\!\!/G)$ restrict, then quotient, surjective (Harada-Landweber, Halpern-Leistner)

K-theory of the projective line

$V = \mathbb{P}^1$ natural sheaves structure, ideal sheaf of point, skyscraper sheaf related by exact sequence so $(1 - H^{-1}) = [\mathcal{O}_p], p \in \mathbb{P}^1$

$K(\mathbb{P}^1) = \mathbb{Q}[H, H^{-1}]/((1 - H^{-1})^2 = 0)$

Generalizes to any toric variety (Vistoli-Vezzosi). Replace Chern classes H_j of tautological bundles by *Grothendieck-Chern classes* $1 - H_j^{-1}$.

Quantum product in cohomology

Push-pull over moduli spaces of 3-marked 0-genus stable maps $\overline{\mathcal{M}}_{0,3}(X,d)$ of class $d \in H_2(X)$:

$\mathrm{ev}_j : \overline{\mathcal{M}}_{0,3}(X,d) \to X$ evaluation at j-th marking

$a \star b = \sum_{\textit{curve classes } d} q^d [\mathrm{ev}_{3,*}(\mathrm{ev}_1^* a \cup \mathrm{ev}_2^* b)]$

Quantum product in cohomology

In quantum K-theory there is a correction to $\mathrm{ev}_{3,*}$ coming from the non-standard inner product. (Givental,Lee)

Because the degree axiom is missing, the products can be infinite. But they are conjectured/known to be finite in many cases.

One can also twist the theory by a "level" ℓ : tensor power K^ℓ of the canonical bundle K (Ruan-Zhang)

I will assume that I am in some situation where fractional powers make sense. For example, if Riemann-Roch holds then one can take fractional tensor powers of line bundles in K-theory using the inverse of the Chern character.

Quantum Kirwan map

Thm (with Gonzalez) There are (formal) maps $Q\kappa : QH_G(V) \to QH(V/\!\!/G), QK_G(V) \to QK(V/\!\!/G)$ quantizing the Kirwan map whose linearization

$D_\alpha Q\kappa : T_\alpha QK_G(V) \to T_{QK(\alpha)}(QK(V/\!\!/G))$ is a homomorphism.

These can be used to give presentations of the quantum cohomology and quantum K-theory rings for toric varieties (at $QK(\alpha)$); the kernel is the Batyrev ideal.

Gauged affine maps

General quantum philosophy: to relate two "quantum rings" need a moduli space that admits two kinds of evaluation maps. Already present in proof of deformation invariance of quantum cohomology.

$\mathcal{M}_n^G(A, V)$ is a moduli space of maps to the quotient stack V/G with domain a curve equipped with a differential $\lambda = c\mathrm{d}z$ with a double pole.

Complement of pole has affine structure, so that the pole maps to $V/\!\!/G$. Naturally compactifies with λ becoming 0 or ∞ on some components.

Stability condition: No automorphisms. So $\lambda = 0$ or ∞ requires three special points, otherwise two.

Stratification by combinatorial type corresponding to *colored trees* of Boardman-Vogt, as opposed to *trees* appearing in stable maps.

Evaluation maps $\mathrm{ev}_j : \overline{\mathcal{M}}_n^G(A, V) \to V/G, j = 1, \ldots, n$
$\mathrm{ev}_{0,d} : \overline{\mathcal{M}}_n^G(A, V, d) \to V/\!\!/G.$

Roughly speaking:

$QK(a) = \sum_{curve\ classes\ d} q^d [\mathrm{ev}_{0,d,*}(\mathrm{ev}_1^* a)]$

correction in quantum K-theory from inner product.

RING HOMOMORPHISM

$QK(a \star b) = QK(a) \star QK(b)$ follows from a divisor class relation in $\overline{\mathcal{M}}_2(A)$.

Bulk deformation

In general there are "additional bubbles" which contribute a "bulk deformation". The divisor classes get corrected by some enumerative formula (which is not that easy to compute.)

This additional correction is the "mirror map" or "relation between GLSM and NLSM" in the physics language

EXAMPLE: QH AND QK OF TORIC VARIETIES

Quantum cohomology of toric varieties: Let $\mu_1, \ldots, \mu_k \in \mathfrak{g}^\vee$ be the weights.

Given a curve class d, define the quantum relations

$$\prod_{\langle \mu_j, d\rangle \geq 0} D_j^{\langle \mu_j, d\rangle} - q^d \prod_{\langle \mu_j, d\rangle \leq 0} D_j^{-\langle \mu_j, d\rangle} = 0.$$

Example: $\mathbb{P}^k$ has relations $D_1 \ldots . D_{k+1} = q$.

Example: QH and QK of toric varieties

G torus, V vector space, $M_1^G(A, V, d)$ is polynomials $(u_1, \ldots, u_k)$ with degree d, semistable at infinity.

Proof of Batyrev relation $D_1 ... D_k = q$ for $V /\!\!/ G = \mathbb{P}^{k-1}$:

$M_1^G(A, V, d)$ tuples of polynomials $(u_1, \ldots, u_k)$ of at most/ with at least one of degree d.

$D_1 ... D_k$ is the Euler class of $\mathbb{C}^k$

Take the section given by evaluation at z_1

Divide by z to get $M_1^G(A, V, d-1)$.

Integral of $D_1, \ldots, D_k$ over $M_1^G(A, V, 1)$ is the integral of 1 over $M_1^G(A, V, 0)$ which is the identity.

Count of relations via MMP

Combinatorics: $QH_G(V)$/Batyrev ideal is the ring of functions on the critical set of a "potential function" , which in the case of projective space is

$W(y_1, \ldots, y_n) = y_1 + \ldots + y_n + q/(y_1 \ldots . y_n)$.

$Crit(W) = \{y_1 = \ldots = y_n, y_1^{n+1} = q\}$.

Claim: $\dim(QH((V/\!\!/G))$ equals the number of critical points of W equals $\dim QH(V)/$Batyrev ring.

In the Fano case, this is an easy consequence of Kouchnirenko's theorem: $\#Crit(W) = n!\,\mathrm{Vol}(P^\vee)$.

Each vertex of P contributes 1 to this formula.

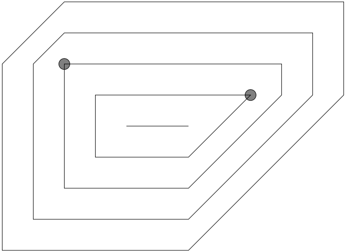
But by Bialincki-Birula decomposition, $\dim(QH(V/\!\!/G))$ is the number of vertices of P.

PROOF OF KOUCHNIRENKO'S THEOREM

(Atiyah) $\#Crit(W)$ is the degree of the hypersurface $\mathcal{O}_{X^\vee}(1)$, which is the symplectic volume times $n!$.

On the other hand, the moment map identifies this integral as the volume of $P^\vee$ times $n!$.

The non-Fano case reduces to the Fano case by the minimal model program.



One still has to do an induction (as in the classical case) but it's more conceptual.

Quantum K-theory of toric varieties

Same argument works in QK: The Batyrev relations

$$\prod_{\langle \mu_j, d\rangle \geq 0} (H_j^{\ell}(1 - H_j^{-1}))^{\langle \mu_j, d\rangle} - q^d \prod_{\langle \mu_j, d\rangle \leq 0} (H_j^{\ell}(1 - H_j^{-1}))^{-\langle \mu_j, d\rangle} = 0.$$

hold but there can be quantum corrections to the generators even in the Fano case. (Chern classes replaced by shifted Grothendieck classes) The inner product should have a residue formula.

The example of Grassmannians

Let $X = \mathrm{Gr}(k, n) = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) /\!\!/ SL(k) = V /\!\!/ G$

Then $QH(X)$ is a quotient of $H(BG) = Sym(\mathfrak{g}^\vee)^G$.

Maps to V/G of class $(d_1, \ldots, d_k)$ are tuples $u_1(z), \ldots, u_k(z)$ of polynomials.

$U = \mathbb{C}^k, Q = \mathbb{C}^{n-k}$ descend to quotient, universal bundles.

Integrating $\mathrm{ev}_1^* c_{n-k+1}(Q \oplus U)$ corresponds to taking the locus of dependence $(s_1 \wedge \ldots \wedge s_k)^{-1}(0)$ of k sections $s_1 \wedge \ldots \wedge s_k$.

Originally we tried taking the zeroth derivatives at the markings. This didn't work.

This week I tried a different method which seems to work. Take the top derivative at the marking z_1. After removing excess intersection, get a $\mathbb{P}^{k-1}$ bundle

$(s_1 \wedge \ldots \wedge s_k)^{-1}(0) \to \overline{\mathcal{M}}_1^G(\mathbb{C}, V, d-1)$

Get the relations $c(Q)c(U) = q$ in QH

$(H_1 \ldots H_n)^{\ell k} g(Q) g(U) = q$ in QK in good range $\ell \in [-1, 0)$, which is what Ming told me last year.

BIRATIONAL INVARIANCE

Recently Kontsevich pointed out that expected properties of birational behavior of QH implies non-rationality of very general cubic four-folds.

It's not clear if QK is really invariant under flops, or behaves well under the mmp.

Thm: QK is invariant under "simple flips" of git type:

Suppose we vary ω so that $V /\!\!/ G$ passes through a "wall" with equal number of weights ± 1. Then $QK(V /\!\!/ G)$ is invariant.

Speculation: At the moment we're looking at the wrong invariant. Maybe combine all levels in some way to get something that has good properties under flips/flops?