Stokes matrices, character varieties, and points on spheres
(with Junho Peter Whang)

Yu-Wei Fan

University of California, Berkeley

Algebra and Algebraic Geometry seminar at UBC
February 2021
Upshots:

1. Establish connections among braid group actions and natural braid group invariants on the following spaces:
   - $SL_2(\mathbb{C})$-character varieties of certain Riemann surfaces,
   - space of Stokes matrices,
   - moduli of points on spheres.

2. Obtain certain finiteness results for integral Stokes matrices, via arithmetic dynamics results on character varieties.
Stokes matrices and exceptional collections

- **Stokes matrices** $V(r) = \{\text{unipotent upper triangular } r \times r \text{ matrix}\}$.
- **Exceptional collections** of triangulated categories are natural sources of examples of integral Stokes matrices.

Let $\mathcal{D}$ be a triangulated category. A collection $\mathcal{E} = \{E_1, \ldots, E_r\}$ is called exceptional if
  - $\text{Hom}^\bullet(E_i, E_i) = \mathbb{C}[0]$ for all $i$,
  - $\text{Hom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Its Gram matrix $(\chi(E_i, E_j))_{1 \leq i, j \leq r}$ is a Stokes matrix, where

$$\chi(E_i, E_j) = \sum_k (-1)^k \dim \text{Hom}^k(E_i, E_j).$$
Braid group actions on $V(r)$

Braid group on $r$-strands:

$$B_r = \langle \sigma_1, \ldots, \sigma_{r-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2 \rangle .$$

$B_r$-actions:

- On $V(r)$:

$$s \xrightarrow{\sigma_i} \begin{bmatrix} \mathbb{I}_{i-1} & s_{i,i+1} & -1 \\ 1 & 0 & \mathbb{I}_{r-i-1} \end{bmatrix} \begin{bmatrix} \mathbb{I}_{i-1} \\ s_{i,i+1} & 1 \\ -1 & 0 & \mathbb{I}_{r-i-1} \end{bmatrix} .$$

- Categorical level: (left) mutations

$$\{E_1, \ldots, E_r\} \xrightarrow{\sigma_i} \{E_1, \ldots, E_{i-1}, \mathbb{L}_i E_{i+1}, E_i, E_{i+2}, \ldots, E_r\} .$$

$$\mathbb{L}_E F \rightarrow \text{Hom}^\bullet (E, F) \otimes E \rightarrow F \xrightarrow{[1]}$$
Braid group invariants on \( V(r) \)

- It’s not hard to check that the conjugacy class of \( -s^{-1}s^T \) is invariant under the \( B_r \)-action on \( V(r) \).
- Let \( \mathcal{E} = \{E_1, \ldots, E_r\} \) be a full exceptional collection of \( \mathcal{D} \), and denote \( s_\mathcal{E} \) its Gram matrix. Then the induced automorphism of \( \text{Serre}_\mathcal{D} \) on \( K_0^{\text{num}}(\mathcal{D}) \) is given by \( s^{-1}_\mathcal{E} s^T_\mathcal{E} \), with respect to the basis \( \{[E_1], \ldots, [E_r]\} \).

**Problem:** Classify full exceptional collections of \( \mathcal{D} \) up to mutations?

**Problem:** Classify integral Stokes matrices up to \( B_r \)-actions in

\[
V_p(r) = \{ s : \det(\lambda + s^{-1}s^T) = p(\lambda) \} \subseteq V(r)\
\]

**Results:**

- \( r = 3 \): when \( \text{disc}(p) \neq 0 \), \( V_p(3) \) contains at most finitely many integral \( B_3 \)-orbits (can be proved by Markoff descent argument)
- \( r = 4 \): when \( \text{disc}(p) \neq 0 \), \( V_p(4) \) contains at most finitely many integral \( B_4 \)-orbits (F.–Whang)
Markoff numbers

A Markoff number is a positive integer $x, y$ or $z$ that satisfies the Diophantine equation

$$x^2 + y^2 + z^2 = xyz.$$ 

- 3 is a Markov number since $(x, y, z) = (3, 3, 3)$ is a solution.
- Vieta involution: $(x, y, z) \rightarrow (yz - x, y, z)$.
- First few Markov numbers: $3, 6, 15, 39, 87, \ldots$. 
3 × 3 Stokes matrices and Markoff-type equation

\[
\begin{bmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{bmatrix}
\]

The characteristic polynomial of \(-s^{-1}s^T\) is given by

\[
p_k(\lambda) = (\lambda + 1)(\lambda^2 - k\lambda + 1),
\]

where

\[
k = x^2 + y^2 + z^2 - xyz - 2.
\]

The braid group \(B_3\)-action on \(V(3)\) gives the Vieta involution,

\[
e.g.: (x, y, z) \mapsto (x, z, xz - y).
\]
Markoff-type equation and one-holed torus

Vogt–Fricke: $\text{Hom}(\pi_1(\Sigma_{1,1}), \text{SL}_2(\mathbb{C})) \parallel \text{SL}_2(\mathbb{C}) \sim \mathbb{C}^3_{x,y,z}$, with coordinates given by

$$x = \text{tr}(\rho(a)), \quad y = \text{tr}(\rho(b)), \quad z = \text{tr}(\rho(ab)).$$

The boundary trace is

$$k := \text{tr}(\rho(d)) = x^2 + y^2 + z^2 - xyz - 2.$$
Vieta involutions and Dehn twists

Dehn twist along $a$ acts by $\tau_a^* : (x,y,z) \mapsto (x,z,xz-y)$, since $\text{tr}(A^2B) = \text{tr}(A)\text{tr}(AB) - \text{tr}(B)$ for any $A, B \in SL_2$.

Similarly, the Dehn twists along $b$ and $ab$ correspond to the other two Vieta involutions.

$B_3 = \langle \tau_a, \tau_b | \tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b \rangle$ form a braid group, and the boundary trace $k$ is a $B_3$-invariant.
We have seen a connection between

- one-holed torus $\leftrightarrow$ 3 $\times$ 3 Stokes matrix
- Dehn twists $\leftrightarrow$ mutations ($B_3$-actions)
- boundary trace $\leftrightarrow$ $-s^{-1}s^T$ ($B_3$-invariants)

$$X_k(\Sigma_{1,1}, SL_2(\mathbb{C})) \cong V_{p_k}(3)$$

Here

$$X_k(\Sigma_{1,1}, SL_2) = \{\rho \in X(\Sigma_{1,1}, SL_2) : \text{tr}\rho(d) = k\}$$

and

$$V_{p_k}(3) = \{s \in V(3) : \text{det}(\lambda + s^{-1}s^T) = p_k(\lambda)\}$$

are $B_3$-invariant subvarieties.

Can we generalize these?
Let $\rho \in X = \text{Hom}(\pi_1(\Sigma_{1,2}), \text{SL}_2(\mathbb{C}))$, and define

\[
\begin{align*}
    a &= \text{tr} \rho(\alpha_1), & b &= \text{tr} \rho(\alpha_2), & c &= \text{tr} \rho(\alpha_3), \\
    d &= \text{tr} \rho(\alpha_1 \alpha_2 \alpha_3), & e &= \text{tr} \rho(\alpha_1 \alpha_2), & f &= \text{tr} \rho(\alpha_2 \alpha_3).
\end{align*}
\]

$X_{k_1,k_2}(\Sigma_{1,2}, \text{SL}_2(\mathbb{C}))$ with fixed boundary traces $k_1 := \text{tr} \rho(\beta_1)$ and $k_2 := \text{tr} \rho(\beta_2)$ is a 4-dimensional subvariety of $\mathbb{C}^6_{a,b,c,d,e,f}$ defined by

\[
\begin{align*}
    ac + bd - ef &= k_1 + k_2 \\
    a^2 + b^2 + \cdots + f^2 - abe - adf - bcf - cde + abcd - 4 &= k_1 k_2
\end{align*}
\]
4 × 4 Stokes matrices

\[ s = \begin{bmatrix}
1 & a & e & d \\
0 & 1 & b & f \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1
\end{bmatrix} \]

The characteristic polynomial of \(-s^{-1}s^T\) is given by

\[ p_{k_1,k_2}(\lambda) = \lambda^4 - k_1 k_2 \lambda^3 + (k_1^2 + k_2^2 - 2)\lambda^2 - k_1 k_2 \lambda + 1 \]

where

\[ k_1 + k_2 = ac + bd - ef, \]

\[ k_1 k_2 = a^2 + b^2 + \cdots + f^2 - a b e - a d f - b c f - c d e + a b c d - 4. \]
Question:

\[
\begin{array}{c c c c}
\text{Space} & \text{SL}_2(\mathbb{C})\text{-character varieties of surfaces} & \uparrow & \text{Braid invariants} \\
& & \text{boundary monodromy} & \\
& \downarrow & \text{space of Stokes matrices} & -s^{-1}s^T
\end{array}
\]
The goal of this talk is to explain the following picture:

<table>
<thead>
<tr>
<th>Space</th>
<th>Braid invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}_2(\mathbb{C})$-character varieties of surfaces</td>
<td>boundary monodromy</td>
</tr>
<tr>
<td>$\parallel$</td>
<td>$\parallel$</td>
</tr>
<tr>
<td>moduli space of points on complex affine 3-sphere</td>
<td>Coxeter invariants</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>space of Stokes matrices</td>
<td>$-s^{-1}s^T$</td>
</tr>
</tbody>
</table>

Next, we’ll introduce the moduli space of points on spheres, braid group actions on them, and natural braid group invariants (the Coxeter invariants).
Moduli space of points on spheres

- \( S(m) := \) complex affine hypersurface in \( \mathbb{C}^m \) defined by
  \[
  \langle x, x \rangle = x_1^2 + \cdots + x_m^2 = 1.
  \]

- Define the moduli space of \( r \) points on \( S(m) \) to be
  \[
  A(r, m) := S(m)^r \sslash SO(m)
  \]
  where \( SO(m) \) acts diagonally.
Quandles and braid group actions

Definition

A quandle is a set $X$ and a binary operation $\triangleleft : X \times X \to X$ such that:
- $x \triangleleft x = x$ and $x \triangleleft - : X \to X$ is a bijection for any $x \in X$.
- $x \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft (x \triangleleft z)$ for any $x, y, z \in X$.

If $(X, \triangleleft)$ is a quandle, then $B_r$ acts on $X^r$ by the following moves:

$$
\sigma_i(x_1, \ldots, x_r) = (x_1, \ldots, x_{i-1}, x_i \triangleleft x_{i+1}, x_i, x_{i+2}, \ldots, x_r), \quad 1 \leq i \leq r - 1.
$$

Example: $r = 3$.

$$
\sigma_1 \sigma_2 \sigma_1(x_1, x_2, x_3) = ((x_1 \triangleleft x_2) \triangleleft (x_1 \triangleleft x_3), x_1 \triangleleft x_2, x_1),
\sigma_2 \sigma_1 \sigma_2(x_1, x_2, x_3) = (x_1 \triangleleft (x_2 \triangleleft x_3), x_1 \triangleleft x_2, x_1).
$$
Examples of quandles

Example (Group quandle)
Let $G$ be a group. Then $u \triangleleft v := uv^{-1}u$ gives a quandle structure on $G$.

Example (Sphere quandle)
For any $u \in S(m)$, define $s_u \in O(m)$ by $s_u(v) := 2 \langle u, v \rangle u - v$. Then $u \triangleleft v := s_u(v)$ gives a quandle structure on $S(m)$.

It’s easy to check that the orthogonal group $O(m)$ acts on $S(m)$ by quandle automorphisms. Therefore the $B_r$-action on $S(m)^r$ naturally descends to a $B_r$-action on the moduli space of points on spheres

$$A(r, m) = S(m)^r \sslash SO(m).$$
Clifford algebras

Let $q$ be a non-degenerate quadratic form over $V/\mathbb{C}$. Then the Clifford algebra is

$$\Cl(V, q) := \bigoplus_{i=0}^{\infty} V^\otimes i / \langle v \otimes v - q(v) : v \in V \rangle$$

In the Clifford algebra:

- For $u \in S(q)$, we have $u^\otimes 2 = q(u) = 1$.
- For $u, v \in S(q)$, we have $s_u(v) = u \otimes v \otimes u^{-1}$.

The Pin group $\Pin(q)$ is the closed algebraic subgroup of $\Cl(q)^\times$ generated by $S(q)$. 
Coxeter invariants of $A(r, q)$

Consider the map

$$c : S(q)^r \to \text{Pin}(q), \quad (u_1, \ldots, u_r) \mapsto u_1 \otimes \cdots \otimes u_r.$$ 

It is $B_r$-invariant:

$$c(\sigma_i u) = u_1 \otimes \cdots \otimes s_{u_i}(u_{i+1}) \otimes u_i \otimes u_{i+2} \otimes \cdots \otimes u_r$$

$$= u_1 \otimes \cdots \otimes (u_i \otimes u_{i+1} \otimes u_i^{-1}) \otimes u_i \otimes u_{i+2} \otimes \cdots \otimes u_r$$

$$= u_1 \otimes \cdots \otimes u_r = c(u).$$

- The Coxeter invariant of $A(r, q)$ to be the $B_r$-invariant morphism

$$c : A(r, q) \to \text{Pin}(q) / \text{SO}(q), \quad [u_1, \ldots, u_r] \mapsto [u_1 \otimes \cdots \otimes u_r]$$

- For each $P \in \text{Pin}(q) / \text{SO}(q)$, denote

$$A_P(r, q) := c^{-1}(P).$$
Exceptional isomorphisms

Theorem (F.–Whang, 2020)

Write \( r = 2g + n \geq 3 \) where \( n \in \{1, 2\} \). We have a \( B_r \)-equivariant isomorphism

\[
A_P(r, 4) \cong X_k(\Sigma_{g,n}, \text{SL}_2(\mathbb{C})),
\]

where the Coxeter invariant \( P \) determine the boundary monodromy \( k \), and vice versa.

Note: The definitions of \( A(r, m) \) and their Coxeter invariants have nothing to do with any Riemann surface!

Corollary (F.–Whang, 2020)

The \( B_r \)-action on \( A_P(r, 4) \) extends to an action of the pure mapping class group of \( \Sigma_{g,n} \).

Similar statements hold for \( A(r, 1) \) and \( A(r, 2) \). The proof relies on the existence of group structure for spheres of dimension 0, 1, 3.
Braid actions and boundary curves of $\Sigma_{g,n}$

$r = 2g + n$, where $n \in \{1, 2\}$.

- $\pi_1(\Sigma_{g,n})$ is a free group of rank $2g + n - 1 = r - 1$.
- Dehn twists along $\alpha_1, \ldots, \alpha_{r-1}$ generate the braid group $B_r$.
- The boundary curve(s) are homotopic to:
  - $r$ odd: $(\alpha_1 \alpha_3 \cdots \alpha_{r-2})(\alpha_1 \alpha_2 \cdots \alpha_{r-1})^{-1}(\alpha_2 \alpha_4 \cdots \alpha_{r-1})$.
  - $r$ even: $\alpha_1 \alpha_3 \cdots \alpha_{r-1}$ and $(\alpha_1 \alpha_2 \cdots \alpha_{r-1})^{-1}(\alpha_2 \alpha_4 \cdots \alpha_{r-2})$.
- $B_r$ acts on $X(\Sigma_{g,n}, G)$ via Dehn twists, and the conjugacy classes of the boundary curve(s) give $B_r$-invariants.
Sketch of proof of main theorem

Identifying the spaces:

- Over $\mathbb{C}$, the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2 \sim x_1x_4 - x_2x_3$.
- $S(4) \cong \text{SL}_2$ carries an action of $\text{SL}_2 \times \text{SL}_2$ by $(g_1, g_2) \cdot u = g_1ug_2^{-1}$.
- We have isomorphisms

$$X(\Sigma_{g,n}.\text{SL}_2) \cong \text{SL}_2^r \parallel (\text{SL}_2 \times \text{SL}_2) \cong S(4)^r \parallel \text{SO}(4) = A(r, 4)$$

$$\rho \mapsto [\rho(\alpha_1\alpha_2\cdots\alpha_{r-1}), \rho(\alpha_2\cdots\alpha_{r-1}), \ldots, \rho(\alpha_{r-1}), 1].$$

Braid group equivariance of the isomorphisms:

- Dehn twists along $\alpha_1, \ldots, \alpha_{r-1}$ are compatible with $B_r$-actions on $\text{SL}_2^r \parallel (\text{SL}_2 \times \text{SL}_2)$ induced by group quandle of $\text{SL}_2$.
- $uv^{-1}u = s_u(v)$ for any $u, v \in S(4) \cong \text{SL}_2$. 
Match boundary monodromy with Coxeter invariants:

- \( \text{Pin}(4) \cong \text{SL}_2^2 \sqcup \text{SL}_2^2 \) with \( \iota^2 = 1 \) and \( (g_1, g_2)\iota = \iota(g_2, g_1) \).
  
  In particular, the image of \( u \in S(4) \) is \( (u, u^{-1})\iota \).

- \( \text{Pin}(4) \parallel \text{SO}(4) \cong \mathcal{W}(\text{SL}_2)^2 \parallel \mathcal{W}(\text{SL}_2) \), where \( \mathcal{W}(\text{SL}_2) = \text{SL}_2 \parallel \text{SL}_2 \).
  
  \begin{itemize}
    \item \( r \) odd: \( [u_1 \otimes \cdots \otimes u_r] \mapsto [u_1 u_2^{-1} \cdots u_r u_1^{-1} u_2 \cdots u_r^{-1}] \).
    \item \( r \) even: \( [u_1 \otimes \cdots \otimes u_r] \mapsto ([u_1 u_2^{-1} \cdots u_r^{-1}], [u_1^{-1} u_2 \cdots u_r]) \).
  \end{itemize}

Example: \( r = 3 \).

\[
\mathcal{C}([\rho(\alpha_1 \alpha_2), \rho(\alpha_2), 1]) = [\rho(\alpha_1 \alpha_2)\rho(\alpha_2)^{-1} 1 \rho(\alpha_1 \alpha_2)^{-1} \rho(\alpha_2) 1^{-1}]
= [\rho(\alpha_1 \alpha_2^{-1} \alpha_1^{-1} \alpha_2)].
\]
Connect with Stokes matrices

**Idea:** Gram matrix of points on sphere is a symmetric matrix with 1’s on the diagonal $\leftrightarrow (s + s^T)/2$.

- Define $V(r, m) \subseteq V(r)$ to be the subset of Stokes matrices such that $\text{rank}(s + s^T) \leq m$.

- By invariant theory for the orthogonal group, there is an isomorphism $S(m)^r \sslash O(m) \cong V(r, m)$ given by:

\[
[(v_1, \ldots, v_r)] \mapsto \begin{bmatrix}
1 & 2 \langle v_1, v_2 \rangle & \cdots & 2 \langle v_1, v_r \rangle \\
0 & 1 & \cdots & 2 \langle v_2, v_r \rangle \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}.
\]

It’s not hard to check that this is a $B_r$-equivariant isomorphism.
Coxeter invariants on $V(r)$ and $-s^{-1}s^T$

- We define the **Coxeter invariant** on $V(r, m)$ to be the composition

  $$c : V(r, m) \xrightarrow{\sim} S(m)^r \sslash O(m) \to \text{Pin}(m) \sslash O(m).$$

- We show that the composition

  $$V(r) \cong S(r)^r \sslash O(r) \to \text{Pin}(r) \sslash O(r) \to O(r) \sslash O(r)$$

  takes a Stokes matrix $s$ to the conjugacy class of $-s^{-1}s^T$.

- Hence we can use the Coxeter invariant of points on spheres to give a **refinement** of the conjugacy class of $-s^{-1}s^T$ of Stokes matrix $s$. 
Character varieties and Stokes matrices

We have a sequence of $B_r$-equivariant morphisms

$$X(\Sigma_{g,n}, \text{SL}_2(\mathbb{C})) \cong S(4)^r \sslash \text{SO}(4) \overset{2:1}{\rightarrow} S(4)^r \sslash \text{O}(4) \cong V(r, 4) \hookrightarrow V(r).$$

Moreover, suppose $s \in V(r)$ is the image of $\rho \in X(\Sigma_{g,n}, \text{SL}_2(\mathbb{C}))$. Then the boundary trace(s) of $\rho$ and the characteristic polynomial $p(\lambda)$ of $-s^{-1}s^T$ are related as follows.

- When $r$ is odd: Let $k$ be the boundary trace of $\rho$, then
  $$p(\lambda) = (\lambda^2 - k\lambda + 1)(\lambda + 1)(\lambda - 1)^{r-3}.$$

- When $r$ is even: Let $k_1, k_2$ be the boundary traces of $\rho$, then
  $$p(\lambda) = \left(\lambda^4 - k_1 k_2 \lambda^3 + (k_1^2 + k_2^2 - 2)\lambda^2 - k_1 k_2 \lambda + 1\right)(\lambda - 1)^{r-4}.$$
Diophantine result on $X_k(\Sigma_{g,n}, \text{SL}_2)(\mathbb{Z})$

- A point $\rho \in X_k(\Sigma_{g,n}, \text{SL}_2)$ is **integral** if its monodromy trace along every essential simple closed curve is integral.

- **Whang, 2017**: The non-degenerate integral points of $X_k(\Sigma_{g,n}, \text{SL}_2)$ consist of finitely many mapping class group orbits.

- The morphism $X_k(\Sigma_{g,n}, \text{SL}_2) \to V_p(r)$ takes an integral point to an integral Stokes matrix:

$$
\rho \mapsto \begin{bmatrix}
1 & \text{tr}\rho(\alpha_1) & \text{tr}\rho(\alpha_1\alpha_2) & \cdots & \text{tr}\rho(\alpha_1\cdots\alpha_{r-1}) \\
1 & \text{tr}(\alpha_2) & \cdots & \text{tr}\rho(\alpha_2\cdots\alpha_{r-1}) \\
1 & \cdots & \text{tr}\rho(\alpha_3\cdots\alpha_{r-1}) \\
\vdots & \ddots & \vdots \\
1 & & & & 1
\end{bmatrix}.
$$
Application to Stokes matrices and exceptional collections

Corollary (F.–Whang, 2020)

Let $p$ be a degree 4 polynomial with $\text{disc}(p) \neq 0$. Then $V_p(4)$ contains at most finitely many integral $B_4$-orbits.

Sketch of proof: By the main theorem, there is a decomposition

$$V_p(4) \cong \bigsqcup_{P \in \text{Pin}(4)//\text{SO}(4)} A_P(4, 4) \cong \bigsqcup_k X_k(\Sigma_{1,2}, \text{SL}_2),$$

and the $B_4$-action coincides with the $\Gamma(\Sigma_{1,2})$-action. The corollary follows from analysis of degenerate integral points on $X_k(\Sigma_{1,2}, \text{SL}_2)$.

Corollary (F.–Whang, 2020)

Let $\mathcal{D}$ be a triangulated category admitting a full exceptional collection of length 4. Assume that $\text{disc}(\det(\lambda + \text{Serre}^\text{num}_\mathcal{D})) \neq 0$. Then there is a finite list of Stokes matrices of rank four such that, up to mutations, the Gram matrix of any full exceptional collection of $\mathcal{D}$ belongs to this list.
Thank you for your attention!

Reference: